Vertex Distinguishing Edge- and Total-Colorings of Cartesian and other Product Graphs

Jean-Luc Baril*, Hamamache Kheddouci† and Olivier Togni*

Abstract

This paper studies edge- and total-colorings of graphs in which (all or only adjacent) vertices are distinguished by their sets of colors. We provide bounds for the minimum number of colors needed for such colorings for the Cartesian product of graphs along with exact results for generalized hypercubes. We also present general bounds for the direct, strong and lexicographic products.

Keywords : graph, edge-coloring, vertex-distinguishing, adjacent vertex-distinguishing, total coloring, total adjacent vertex-distinguishing, graph products.

1 Introduction

All the graphs we deal with are simple, finite and with no component K_2, where K_n stands for the complete graph of order n. Let $G = (V, E)$, be a graph with vertex set V and edge set E. An edge between vertex x and vertex y is denoted by xy. Let $\Delta(G)$ be the maximum degree of the graph.

A *proper edge-coloring* c is a mapping from E to $\mathbb{N}$ such that edges incident with the same vertex receive distinct values (or colors). For any vertex x of G, let $S(x)$ denote the set of the colors of all edges incident to x (if necessary, we write $S_c(x)$ to indicate which coloring is used). A proper edge coloring is said to be

- *vertex distinguishing (VD)* if $S(x) \neq S(y), \forall x, y \in V, x \neq y$;
- *adjacent vertex distinguishing (AVD)* if $S(x) \neq S(y), \forall xy \in E$.

A total coloring of a graph G is a mapping from $V \cup E$ to $\mathbb{N}$ such that neighboring elements receive distinct colors. For a total coloring, let $S(x)$ be the set of the colors of all edges incident to x plus the color of x: $S(x) = \{c(e)|e = xy\} \cup \{c(x)\}$. A *total adjacent vertex distinguishing (TAVD) coloring* is a total coloring satisfying $S(x) \neq S(y), \forall xy \in E$.

The minimum number of colors among all VD-colorings, AVD-colorings and TAVD-colorings respectively of a graph G will be called the *VD-chromatic index*, *AVD-chromatic index* and *TAVD-chromatic index* denoted by $\chi'_s(G)$, $\chi'_a(G)$ and $\chi''_a(G)$ respectively.

The notation $\chi'(G)$, $\chi(G)$ and $\chi''(G)$ is used to represent respectively the chromatic index, the chromatic number and the total chromatic number of G, as usual. A coloring using the least number of colors with respect to the given constraints will be called a *minimal* coloring.

The notion of VD-coloring was introduced in [BS97] and independently in [ČHS96] where $\chi'_s(G)$ is called the *observability*.

A lower bound for the VD-chromatic index is given by $\pi(G) = \min\{k : \binom{k}{d} \geq n_d$ for $1 \leq d \leq \Delta(G)\}$ where n_d is the number of vertices of degree d. Moreover, it is conjectured in [BS97] that

*LE2I, UMR 5158 CNRS, Université de Bourgogne, BP 47870, 21078 Dijon cedex, France, `{barjl, otogni}@u-bourgogne.fr`

†LIESP, Université Claude Bernard Lyon1, 843, Bd. du 11 novembre 1918, 69622 Villeurbanne Cedex France, `hkheddou@bat710.univ-lyon1.fr`

$\chi'_s(G) \leq \pi(G)+1$ for any graph $G \neq K_2$. This conjecture has been solved for graphs of maximum degree two [BBS02] and for graphs verifying $\Delta(G) \geq \sqrt{2|V(G)|}+4$ and $\delta(G) \geq 5$ (where $\delta(G)$ is the minimum degree of G) [BKLS04]. It was proved in [BHBLW99] that $\chi'_s(G) \leq |V(G)|+1$ and in [BHBLW01] the authors obtained $\chi'_s(G) \leq \Delta(G)+5$ if $\delta(G) \geq \frac{n}{3}$.

The study of AVD-colorings is more recent. In [BGLS07], the authors proved that $\chi'_a(G) \leq 5$ for graphs of maximum degree 3 and $\chi'_a(G) \leq \Delta(G)+2$ for bipartite graphs. In [BKT06], the AVD-chromatic index of multidimensional meshes was determined. AVD-colorings are also studied in [EHW06, GR06] under the name of neighbour-distinguishing edge colorings. The bound $\chi'_a(G) \geq \Delta(G)$ is trivial. Moreover if G contains two adjacent vertices of degree $\Delta(G)$ then $\chi'_a(G) \geq \Delta(G)+1$. The following conjecture was made in [ZLW02]:

Conjecture 1 ([ZLW02]) *Let $G \neq C_5$ be a graph of maximum degree Δ, then*

$$\Delta \leq \chi'_a(G) \leq \Delta+2.$$

In relation with this conjecture and using a probabilistic argument, Hatami proved recently that $\chi'_a(G) \leq \Delta(G)+300$ provided that $\Delta(G) > 10^{20}$.

As remarked in [EHW06, ZCL+06], the AVD-chromatic index of some regular graphs is in relation with their total chromatic number. More precisely, if G is a regular graph with $\chi'_a(G) = \Delta(G)+1$ then $\chi''(G) = \Delta(G)+1$ and the converse also holds. Therefore, some of the results of the present paper about the AVD-chromatic index are also new results about the total chromatic number while some other were already known earlier, see [KM03, ZŽ04].

Total adjacent vertex distinguishing colorings were considered in [ZCL+05, LWZW06] in which the authors conjecture that $\chi''_a(G) \leq \Delta+3$.

In this paper we shall consider VD, AVD and TAVD-colorings of products of graphs (see definitions below), trying to derive bounds for χ'_s, χ'_a and χ''_a of the product of two graphs in term of the value of the same parameter on the factors. In Section 2, we present general bounds for the Cartesian product. As an application, we determine in Sections 3 and 4, the AVD and TAVD-chromatic index of the generalized hypercube and present in Section 5 tight lower and upper bounds for its VD-chromatic index. Section 6 provides bounds for VD, AVD and TAVD-chromatic indices of direct, strong and lexicographic products.

We use the following notation from [IK00] for the standard graph products. Let $G\square H$, $G\times H$, $G\boxtimes H$ and $G\circ H$ be the *Cartesian*, *direct* (also called Krönecker or categorical), *strong* and *lexicographic* product of G and H respectively. The vertex set of any of these products is $V(G)\times V(H)$ and the edge sets are defined below:
$E(G\square H) = \{(a,x)(b,y), ab \in E(G) \text{ and } x=y \text{ or } xy \in E(H) \text{ and } a=b\}$.
$E(G\times H) = \{(a,x)(b,y), ab \in E(G) \text{ and } xy \in E(H)\}$.
$E(G\boxtimes H) = E(G\square H) \cup E(G\times H)$.
$E(G\circ H) = \{(a,x)(b,y), ab \in E(G) \text{ or } a=b \text{ and } xy \in E(H)\}$.

The *d-dimensional generalized hypercube* (also known as Hamming graph) K_n^d is the Cartesian product of the complete graph K_n by itself d times: $K_n^d = K_n\square K_n\square\ldots\square K_n$. The hypercube Q_d is the graph K_2^d.

2 General results for Cartesian products

We first present general results for the VD, AVD and TAVD-coloring of the Cartesian product of graphs.

For AVD-colorings of the Cartesian product of a graph and a path or a cycle, the following results were proved in [BKT06]:

Theorem 1 ([BKT06]) *Let $d \geq 2$ be an integer and let G be a graph of maximum degree $\Delta \leq d-1$. If there exists an AVD-coloring of G with d colors, then*

$$\chi'_a(G\Box P_2) \le d+1,$$

$$\chi'_a(G\Box P_k) \le d+2, \text{ for } k \ge 3.$$

Theorem 2 ([BKT06]) *Let $d \ge 2$ be an integer and let G be a graph of maximum degree $\Delta \le d-1$ and of minimum degree $\delta \ge 2$. If there exists an AVD-coloring of G with d colors, then*

$$\chi'_a(G\Box C_k) \le d+2.$$

The following theorem gives an upper bound on χ'_s and χ'_a for the Cartesian product of general graphs.

Theorem 3 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G\Box H) \le \chi'_s(G) + \chi'_s(H),$$
$$\chi'_a(G\Box H) \le \chi'_a(G) + \chi'_a(H).$$

Proof : Note that the product $G' = G\Box H$ consists of $|V(G)|$ copies of H; and there is a perfect matching between any two copies of H if the corresponding vertices of G are adjacent. By symmetry, G' also contains $|V(H)|$ copies of G. Let c_G and c_H be two minimal VD- (AVD-)colorings of G and H respectively such that the colors of c_G are different from those used by c_H. A VD-coloring (AVD-coloring) c' of G' is obtained as follows: each copy of G in G' is colored by c_G and each copy of H in G' is colored by c_H.

Indeed, let (a,x) be a vertex of G'. We have $S_{c'}((a,x)) = S_{c_G}(a) \cup S_{c_H}(x)$ and so, (a,x) is distinguished from another (adjacent for AVD-coloring) vertex (b,y) because a is distinguished from b in G or x is distinguished from y in H. □

Notice that, despite the proof of the above theorem is quite simple, it allows to find the exact value of the AVD-chromatic index for graphs G and H verifying $\chi'_a(G) = \Delta(G)$ and $\chi'_a(H) = \Delta(H)$ (for instance, this is the case for trees with no two adjacent vertices of maximum degree [ZLW02]).

For the TAVD-chromatic index of the Cartesian product of two graphs, we have the following.

Theorem 4 *Let G and H be two graphs different from K_2 such that $\chi(H) \le \chi''_a(G)$, then*

$$\chi''_a(G\Box H) \le \chi''_a(G) + \chi'_a(H).$$

Proof : Let $G' = G\Box H$, c_G be a minimal TAVD-coloring of G and c_H be a minimal AVD-coloring of H such that the colors of c_G are different from those used by c_H. We also color the vertices of H with the colors $0, 1, \dots, \chi(H)-1$. Let $\alpha = \chi''_a(G)$ and denote by σ_i the permutation on $0, 1, \dots, \alpha-1$ defined by $\sigma_i(k) = (k+i) \bmod \alpha$, for $0 \le k \le \alpha-1$ and $0 \le i \le \chi(H)-1$. By extension, the total coloring $c'_G = \sigma_i(c_G)$ is defined by $c'_G(x) = \sigma_i(c_G(x))\ \forall x \in V(G) \cup E(G)$.

In order to obtain an AVD-coloring c' of G', we first use $\sigma_i(c_G)$ to color each copy G_j of G in G', where i is the color (given by the proper vertex coloring of H) of the vertex corresponding to the copy G_j. Since $\chi(H) \le \alpha$, all permutations σ_i are pairwise different;

A TAVD-coloring c' of $G' = G\Box H$ is obtained as follows: each copy G_j of G in G' is totally colored by $\sigma_i(c_G)$, where i is the color (given by the proper vertex coloring of H) of the vertex of H corresponding to copy G_j and each copy of H in G' is colored by c_H.

Notice that since $\chi(H) \le \alpha$, all permutations σ_i are pairwise different thus the colors assigned to vertices of G' form a proper coloring. Moreover, as for the previous theorem, the fact that c' is a TAVD-coloring is easily shown since the colors on the vertices induce a proper vertex coloring and adjacent vertices are distinguished either by their sets of colors from c_G or by their sets of colors from c_H. □

We now propose two better results in more specific cases for the AVD and TAVD-chromatic indices.

Theorem 5 *Let G be a graph such that the degree of each vertex is relatively prime with $\chi'_a(G)$, and let H be a graph verifying $\chi(H) \leq \chi'_a(G)$ then*

$$\chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H).$$

Proof : Let $G' = G \square H$, c_G be an AVD-coloring of G in $\chi'_a(G)$ colors and γ_H be a proper edge coloring of H in $\chi'(H)$ colors distinct from those used by c_G. We also color the vertices of H with the colors $0, 1, \ldots, \chi(H) - 1$.

Similarly with the proof of the previous theorem, let $\alpha = \chi'_a(G)$ and denote by σ_i the permutation on $0, 1, \ldots, \alpha - 1$ defined by $\sigma_i(k) = (k + i) \bmod \alpha$, for $0 \leq k \leq \alpha - 1$ and $0 \leq i \leq \chi(H) - 1$. Let also $c'_G = \sigma_i(c_G)$ be the coloring defined by $c'_G(e) = \sigma_i(c_G(e))$ $\forall e \in E(G)$.

In order to obtain an AVD-coloring c' of G', we first use $\sigma_i(c_G)$ to color each copy G_j of G in G', where i is the color (given by the proper vertex coloring of H) of the vertex corresponding to the copy G_j.

Since $\chi(H) \leq \alpha$, all permutations σ_i are pairwise different; moreover, if x is a vertex of G, and x_i (resp. x_j) its corresponding vertex in G_i (resp. G_j), then the color set of x_i in G_i is different from the color set of x_j in G_j. Then, we have two cases to consider:

Case 1: $\chi'(H) = \Delta(H)$

In this case, we use the proper coloring γ_H of H to color each copy of H in G'. Let (a, x) and (b, y) be two adjacent vertices of G'. Without loss of generality, we have $S_{c'}((a, x)) = S_{c_G}(a) \cup S_{\gamma_H}(x)$ and $S_{c'}((b, y)) = \sigma_i(S_{c_G}(b)) \cup S_{\gamma_H}(y)$, for some i.

If $x = y$ (i.e. (a, x) and (b, y) are in the same copy of G), then $i = 0$ and $S_{c'}((a, x)) \neq S_{c'}((b, y))$ since $S_{c_G}(a) \neq S_{c_G}(b)$.

If $x \neq y$, as (a, x) and (b, y) are adjacent then so are x and y in H, thus x and y have different colors and so $i \neq 0$. If $S_{\gamma_H}(y) \neq S_{\gamma_H}(x)$, we have immediately $S_{c'}((a, x)) \neq S_{c'}((b, y))$. If $S_{\gamma_H}(y) = S_{\gamma_H}(x)$, we prove that $S_{c_G}(a) \neq \sigma_i(S_{c_G}(a))$. Indeed, by contradiction, let d be the degree of a in G; if $S = S_{c_G}(a) = \sigma_i(S_{c_G}(a)) = \{s_1, \ldots, s_d\}$, then we have modulo $\alpha = \chi'_a(G)$: $s_j + i = s_k \in S$ for each j, $1 \leq j \leq d$. If we sum all equalities, we obtain $d.i = 0 \bmod \alpha$. So, the hypothesis that d and α are relatively prime gives $\sigma_i = \sigma_0 = Id$ which is a contradiction.

Therefore, when $\chi'(H) = \Delta(H)$ we have $\chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H)$.

Case 2: $\chi'(H) = \Delta(H) + 1$

Remark that in this case, for each vertex x of H, there exists (at least) one color j such that $j \notin S_{\gamma_H}(x)$ (the *missing color*). In order to complete the coloring c' of G', we use the proper edge-coloring γ_H to color each copy of H. Then, from the above remark, for each copy G_i of G in G', there is a color that is not used by any of the edges incident with any vertex of G_i. So we modify the coloring $\sigma_i(c_G)$ of each copy G_i of G' by changing the color zero by this missing color. With a similar proof as for the first case, we can show that $\chi'_a(G \square H) \leq \chi'_a(G) + \Delta(H)$. $\square$

Theorem 6 *Let G be a graph such that the degree of each vertex plus one is relatively prime with $\chi''_a(G)$, and let H be a graph verifying $\chi(H) \leq \chi''_a(G)$ then*

$$\chi''_a(G \square H) \leq \chi''_a(G) + \Delta(H).$$

Proof : We modify the previous proof as follow: each AVD-coloring is replaced by a TAVD-coloring; thus $\chi'_a(G)$ is changed in $\chi''_a(G)$; each set of d colors $\{s_1, s_2, \ldots, s_d\}$ is changed in a set of $d + 1$ colors $\{s_1, s_2, \ldots, s_d, s_{d+1}\}$ and the equality $d \cdot i = 0 \bmod \alpha$ becomes $(d + 1) \cdot i = 0 \bmod \alpha$. $\square$

Now, if α is taken to be the smallest prime number greater than $\chi'_a(G)$ (respectively $\chi''_a(G)$) then we obtain the two following corollaries.

Corollary 1 *Let G be a graph and let p be the smallest prime number greater than or equal to $\chi'_a(G)$. If H is a graph verifying $\chi(H) \leq p$ then*

$$\chi'_a(G \square H) \leq p + \Delta(H).$$

Corollary 2 *Let G be a graph and let p be the smallest prime number greater than or equal to $\chi''_a(G)$. If H is a graph verifying $\chi(H) \le p$ then*

$$\chi''_a(G \Box H) \le p + \Delta(H).$$

3 AVD-coloring of the generalized hypercube

In this section, we determine the AVD-chromatic index of the generalized hypercube K_n^d. We first need to compute the AVD-chromatic index of $K_{2p} \Box K_2$.

Lemma 1 *For $p \ge 2$*

$$\chi'_a(K_{2p} \Box K_2) = 2p + 1.$$

Proof : In order to construct the graph $K_{2p} \Box K_2$, we consider two copies K and K' of K_{2p}. Let $V(K) = \{x_0, x_1, \ldots, x_{2p-1}\}$ and let $V(K') = \{x'_0, x'_1, \ldots, x'_{2p-1}\}$. Let c be the coloring of K_{2p} defined by:

$$c(x_i x_j) = i + j \mod (2p+1) \text{ with } 0 \le i < j \le 2p - 1.$$

In the following, each integer is considered modulo $(2p+1)$.

Remark that the equality $2i = i - 1$ has no solution for $i \in [0..2p-1]$, and it is easy to see that the color set $S(x_i)$ of each vertex x_i is exactly $S(x_i) = \{0,\ 1,\ \ldots,\ 2p\} \backslash \{2i,\ i-1\}$.

Moreover, for $i \in [0..2p-1]$, $i - 1 \in [0..2p]/\{2p-1\}$ and $2i \in [0..2p]/\{2p-1\}$. So, the color $2p-1$ is the only one which appears in each set $S(x_i)$. Moreover, if we suppose $i \ne j$ then the two equalities $2j = i-1$ and $2i = j-1$ induce that $3(i+1) = 0$ which is impossible when 3 does not divide $2p+1$. This proves that c is an AVD-coloring when 3 does not divide $2p+1$.

Now we define another coloring c' for the second copy K' of K_{2p} by setting:

$$c'(x'_i x'_j) = \sigma(c(x_i x_j)) \text{ with } 0 \le i, j \le 2p-1,$$

where x'_i is the corresponding vertex of x_i in the second copy K' of K_{2p} and the permutation σ is defined by:

$$\sigma(i-1) = 2i, \text{ with } 1 \le i \le 2p.$$

An illustration of the colorings c and c' is given in Appendix for $p = 3$.

As above, remark that if 3 does not divide $2p+1$ and if $i \ne 2p-1$ then $\sigma^2(i) = i$ has no solution. This proves that σ has no cycle of length two in its decomposition into a product of disjoint cycles and c' is also an AVD-coloring if 3 does not divide $2p+1$.

In this case (3 does not divide $2p+1$), we give the color $2i$ to each edge $x_i x'_i$ in $K_{2p} \Box K_2$. So, in $K_{2p} \Box K_2$, the vertex x_i has no incident edge of color $i-1$, and the vertex x'_i has no incident edge of color $\sigma(2i)$. Since σ has no cycle of length 2 in its decomposition then $\sigma(2i) \ne i - 1$. Consequently, $S(x_i) \ne S(x'_i)$ in $K_{2p} \Box K_2$.

So, we have obtained an AVD-coloring of $K_{2p} \Box K_2$ with $2p+1$ colors when 3 does not divide $2p+1$ (see Figure 1 for an AVD-coloring of $K_4 \Box K_2$).

In the case where 3 divides $2p+1$, we modify the coloring c into an AVD-coloring d as follows:

Recall that σ has a unique cycle of length two in its disjoint cycles decomposition. Let (a, b) be this cycle where $a > b$. We have necessarily $a = 2\frac{2p+1}{3} - 2$ and $b = \frac{2p+1}{3} - 2$ since $\sigma^2(i) = i$ has only one solution. Remark also that $a = 2(b+1) = \sigma(b)$.

We define the coloring d by:

$$\begin{cases} d(x_{a+1+k} x_{2p-1-k}) & = a \quad \text{with } 0 \le k \le 2p-2-a, \\ d(x_{a+1+k} x_{2p-k}) & = a - 1 \quad \text{with } 1 \le k \le 2p-2-a, \end{cases}$$

and in the other cases

$$d(x_i x_j) = c(x_i x_j).$$

This coloring is also an AVD-coloring for K_{2p} with $2p+1$ colors. It is easy to see that the color set $S(x_i)$ of each vertex x_i is exactly $S(x_i) = \{0,\ 1,\ \ldots,\ 2p\}\backslash\{2i,\ i-1\}$ for $i \neq a+1$, $i \neq a/2+p$, and $S(x_{a+1}) = \{0,\ 1,\ \ldots,\ 2p\}\backslash\{b,\ a-1\}$, $S(x_{a/2+p}) = \{0,\ 1,\ \ldots,\ 2p\}\backslash\{a,\ a/2+p-1\}$

Recall that we had previously with the coloring c, $S(x_{a+1}) = \{0,\ 1,\ \ldots,\ 2p\}\backslash\{b,\ a\}$, $S(x_{a/2+p}) = \{0,\ 1,\ \ldots,\ 2p\}\backslash\{a-1,\ a/2+p-1\}$.

As above, excepted the color $2p-1$, all colors appear in at least one color set.

Now, we define another AVD-coloring d' for the second copy of K_{2p} by setting:

$$d'(x'_i x'_j) = \sigma'(d(x_i x_j)) \quad \forall\ 0 \leq i, j \leq 2p-1 \ \text{ with}$$

$$\begin{cases} \sigma'(i) & = \sigma(i) \quad \forall i \neq b \ \text{ and } i \neq a/2+p-1 \\ \sigma'(b) & = a-1 \\ \sigma'(a/2+p-1) & = a \end{cases}$$

An illustration of the colorings d and d' is given in Appendix for $p = 4$.

By construction, σ' has no cycle of length 2 in its decomposition into a product of disjoint cycles. Indeed, we have 'broken' the cycle (a, b) of length 2 in σ.

In $K_{2p}\Box K_2$, for each i, the color sets $S(x_i)$ of x_i and $S(x'_i)$ of x'_i are distinct and verify $S(x'_i) = \sigma'(S(x_i))$. Moreover, as for the previous case, for each i there exists a color which appears neither in $S(x'_i)$ nor in $S(x_i)$. We give this color to the edge $x_i x'_i$ for each vertex x_i of K_{2p}. Thus, we obtain an AVD-coloring for $K_{2p}\Box K_2$ in $2p+1$ colors when 3 divides $2p+1$.

Finally, as $K_{2p}\Box K_2$ is regular of degree $2p$, we obtain $\chi'_a(K_{2p}\Box K_2) \geq 2p+1$ and thus $\chi'_a(K_{2p}\Box K_2) = 2p+1$.

□

Theorem 7 *For any integers $n \geq 2$ and $d \geq 2$,*

$$\chi'_a(K_n^d) = d(n-1)+1.$$

Proof : It is known that $\chi'_a(K_n) = \chi'_s(K_n) = n+1-\epsilon(n)$, where $\epsilon(n) = 1$ for odd n and 0 for even n.

When n is odd, $n = 2p+1$ for some p, we proceed by induction on d. The result is true for $d = 1$. Assume that $\chi'_a(K_{2p+1}^{d-1}) = 2p(d-1)+1$, thus $\chi'_a(K_{2p+1}^{d-1})$ and $\Delta(K_{2p+1}^{d-1}) = 2p(d-1)$ are relatively prime and using Theorem 5 with $G = K_{2p+1}^{d-1}$ and $H = K_{2p+1}$, we have that $\chi'_a(K_{2p+1}^d) = 2pd+1$.

When $n = 2p$ is even, we obtain $d(2p-1)+1 \leq \chi'_a(K_{2p}^d) \leq d(2p-1)+2$. We show that $\chi'_a(K_{2p}^d) = 2p+1$ by induction on d.

When $d = 2$, by Lemma 1, there axists an AVD-coloring of $K_{2p}\Box K_2$ with $2p+1$ colors, and applying Theorem 5 with $G = K_{2p}\Box K_2$ and $H = K_p$, we conclude that $\chi'_a((K_{2p}\Box K_2)\Box K_p) = 3p$. Now, we add $p-1$ perfect matchings to $(K_{2p}\Box K_2)\Box K_p$ in order to obtain K_{2p}^2, all the edges of each perfect matching being colored with a new color. We therefore obtain $\chi'_a(K_{2p}^2) = 4p-1$.

We suppose by induction that $\chi'_a(K_{2p}^d) = d(2p-1)+1$ for $d \geq 2$ and prove that $\chi'_a(K_{2p}^{d+1}) = (d+1)(2p-1)+1$. As $K_{2p}^{d+1} = K_{2p}^d\Box K_{2p}$, K_{2p}^{d+1} contains $2p$ copies of K_{2p}^d. Let $\{x_i^j,\ \ 0 \leq i \leq (2p)^d-1\}$ be the set of vertices of the j^{th} copy $(1 \leq j \leq 2p)$. So, we color each copy with the different AVD-colorings c_j, $1 \leq j \leq 2p$ such that: c_1 is an AVD-coloring of K_{2p}^d in $d(2p-1)+1$ colors, and if $j \geq 2$, we define $c_j(x_i^j x_k^j) = c_1(x_{i+j-1}^j x_{k+j-1}^j)$ where the subscripts are modulo $(2p)^d$. In order to obtain K_{2p}^{d+1}, we add $2p-1$ perfect matchings between the $2p$ copies of K_{2p}^d, all the edges of each perfect matching being colored with a new color. So, $\chi'_a(K_{2p}^{d+1}) = d(2p-1)+1+(2p-1) = (d+1)(2p-1)+1$.

□

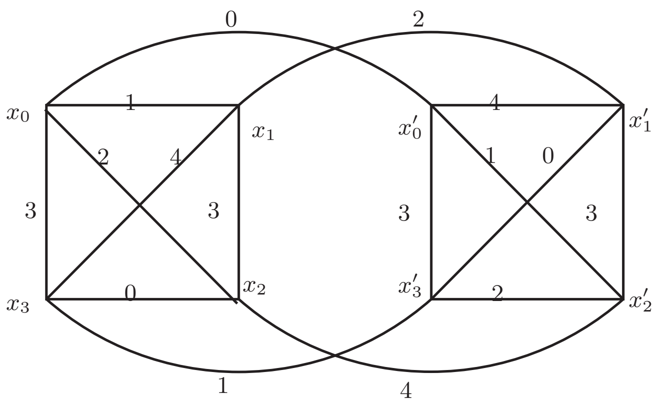

Figure 1: An AVD-coloring for $K_4 \square K_2$ with 5 colors

4 TAVD-coloring of the generalized hypercube

In this section, we determine the TAVD-chromatic index of the generalized hypercube K_n^d. In order to do that, we first need to compute the TAVD-chromatic index of K_{2p+1}^2.

Lemma 2 *For $p \geq 2$*

$$\chi''_a(K_{2p+1}^2) = 4p + 2.$$

Proof : Let $V(K_{2p+1}^2) = \{x_i^j | 0 \leq i, j \leq 2p\}$ and $E(K_{2p+1}^2) = \{x_i^j x_i^{j'} | 0 \leq i, j, j' \leq 2p, j \neq j'\} \cup \{x_i^j x_{i'}^j | 0 \leq i, i', j \leq 2p, i \neq i'\}$. Define a total coloring c of K_{2p+1}^2 by

$$\begin{cases} c(x_i^j) & = & 2i + j \bmod (4p+2), \\ c(x_i^j x_{i'}^j) & = & i + i' + j \bmod (4p+2), \\ c(x_i^j x_i^{j'}) & = & i + j + j' + 2p + 1 \bmod (4p+2). \end{cases}$$

We now show that c is a TAVD-coloring. By the above definition, $S(x_i^j) = \{0, \ldots 4p+1\} \setminus \{i + j + 2p + 1 \bmod (4p+2)\}$. Hence, for $i' \neq i$ and $j' \neq j$, we have $S(x_i^j) \neq S(x_{i'}^j)$ and $S(x_i^j) \neq S(x_i^{j'})$. $\square$

Theorem 8 *For any integers $n \geq 2$ and $d \geq 2$,*

$$\chi''_a(K_n^d) = (n-1)d + 2.$$

Proof : It is known [ZCL+05] that $\chi''_a(K_n) = \chi''_s(K_n) = n + 2 - \epsilon(n)$, where $\epsilon(n) = 0$ for odd n and 1 for even n.

We proceed by induction on d, by considering two cases depending on the parity of n.

When n is even, $n = 2p$ for some p, the result is true for $d = 1$. Assume that $\chi''_a(K_{2p}^{d-1}) = (2p-1)(d-1) + 2$, thus $\chi''_a(K_{2p}^{d-1})$ and $\Delta(K_{2p}^{d-1}) + 1 = (2p-1)(d-1) + 1$ are relatively prime and using Theorem 6 with $G = K_{2p}^{d-1}$ and $H = K_{2p}$, we have that $\chi''_a(K_{2p}^d) = (2p-1)d + 2$.

When $n = 2p+1$ is odd, the result is true for $d = 2$ by Lemma 2. Assume that $\chi''_a(K_{2p+1}^d) = 2pd + 2$. Thus $\chi''_a(K_{2p+1}^{d-1})$ and $\Delta(K_{2p+1}^{d-1}) + 1 = 2pd + 1$ are relatively prime and using Theorem 6 with $G = K_{2p+1}^{d-1}$ and $H = K_{2p+1}$, we have that $\chi''_a(K_{2p+1}^d) = 2pd + 2$.

Therefore, we have proved that $\chi''_a(K_n^d) = (n-1)d + 2$.

$\square$

5 VD-coloring of the generalized hypercube

Finding a minimal VD-coloring for the product of a graph by K_2 seems to be difficult, nevertheless we present two simple upper bounds.

Theorem 9 *For any graph G, $\chi'_s(G \square K_2) \leq 2\chi'_s(G) + 1$.*

Proof : A VD-coloring of $G \square K_2$ can be simply obtained by coloring the edges of each of the two copies of G with two VD-colorings using $\chi'_s(G)$ colors distinct from each other and by coloring the edges between the two copies by a new color. □

Given a graph G with an edge-coloring c, we say that a color j *touches* a vertex x if $j \in S_c(x)$.

Theorem 10 *If there exists a VD-coloring of G with d colors such that one color touches each vertex, then there exists a VD-coloring of $G \square K_2$ with $d+2$ colors such that one of them touches each vertex.*

Proof : Assume that c is a VD-coloring of G with colors $\{0, 1, \ldots, d-1\}$ such that the color 0 touches each vertex of G. Color each of the two copies of G in $G \square K_2$ with the coloring c and replace the color 0 by the color d in the first copy and by the color $d+1$ in the second copy. Now, give the color 0 to the edges of the perfect matching between the two copies. The coloring obtained is clearly VD since the coloring in each copy is VD and a vertex of a copy is distinguished with a vertex of the other copy since different colors touch them. Moreover, the color 0 touches each vertex of $G \square K_2$. □

This theorem allows to obtain the known fact that $\chi'_s(Q_n) \leq 2n$ (see [ČHS96]). Notice that the authors of [ČHS96] have given the asymptotic value of $\chi'_s(Q_n)$ but finding the exact value still remains an open problem.

Theorem 11 *For any integers $n \geq 3$ and d,*

$$d(n-1) \leq \chi'_s(K_n^d) \leq d(2\lfloor \frac{n}{2} \rfloor + 1).$$

Proof : Obviously, $\chi'_s(K_n^d) \geq \Delta(K_n^d) + 1 = d(n-1) + 1$. The relation is true for $d = 1$. By recurrence on d and with Theorem 3, when $n = 2p+1$, $\chi'_s(K_{2p+1}^d) \leq \chi'_s(K_{2p+1}^{d-1}) + \chi'_s(K_{2p+1}) \leq d(2p+1)$.

The case $n = 2p$ is similar.

□

Theorem 12 *For any integers $p \geq 1$ and $d \geq 1$,*

- *if $d \leq \ln(2p+1) + 1$ then $\chi'_s(K_{2p+1}^d) = d(2p+1)$,*
- *if $d \leq \ln(2p)$ then $\chi'_s(K_{2p}^d) \geq 2dp - 1$.*

Proof :

For the first assertion, in order to see if $d(2p+1)$ colors are sufficient to obtain a VD-coloring, we compare $\binom{(2p+1)d-1}{2dp} = \binom{(2p+1)d-1}{d-1}$ with $(2p+1)^d$. Let us define:

$$A = \frac{\binom{(2p+1)d-1}{d-1}}{(2p+1)^{d-1}} = \frac{((2p+1)d-1)((2p+1)d-2)\ldots(2dp+1)}{(d-1)(2p+1)(d-2)(2p+1)\ldots 1(2p+1)}.$$

Thus $\ln(A) = \sum\limits_{\ell=1}^{d-1} \ln(\frac{2dp+\ell}{\ell(2p+1)}) = \sum\limits_{\ell=1}^{d-1} \ln(\frac{2dp+\ell}{2p+1}) - \sum\limits_{\ell=1}^{d-1} \ln(\ell)$.

With the well-known Darboux sums inequalities, we obtain

$$
\begin{aligned}
\ln(A) &\leq \int_{\ell=1}^{d} \ln(\tfrac{2dp+\ell}{2p+1})d\ell - \int_{\ell=1}^{d-1} \ln(\ell)d\ell \text{ and (with Maple),}\\
&\leq (1+2dp)\ln(1+\tfrac{d-1}{2dp+1}) + (d-1)\ln(\tfrac{d}{d-1}) - 1 \text{ and (with } ln(1+x) < x \text{ for } x \neq 0),\\
&< d-1.
\end{aligned}
$$

So, if $d \leq \ln(2p+1)+1$ then $\ln(A) < \ln(2p+1)$ and $A < 2p+1$. Thus when $d \leq \ln(2p+1)+1$, $\chi'_s(K^d_{2p+1}) \geq d(2p+1)$. With the previous theorem, we obtain the result.

The proof of the second point is similar. □

6 Other products

We first present a simple fact about TAVD-colorings that will be useful.

Fact 1 *For any graph G,*

$$\chi''_a(G) \leq \chi'(G) + \chi(G).$$

Actually, coloring separately the edges and the vertices of the graph gives a proper total coloring and the colors assigned to vertices of G distinguish adjacent vertices.

As a consequence, the TAVD-chromatic index of a bipartite graph B with two adjacent vertices of maximum degree is $\Delta(B)+2$ since $\Delta(B)+2$ colors are necessary to color and distinguish two adjacent vertices of degree $\Delta(B)$.

6.1 Direct product

Theorem 13 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \times H) \leq \chi'_s(G)\chi'_s(H),$$

$$\chi'_a(G \times H) \leq \min\{\chi'(G)\chi'_a(H), \chi'_a(G)\chi'(H)\},$$

$$\chi''_a(G \times H) \leq \min\{\chi'(G)\chi''_a(H), \chi''_a(G)\chi'(H)\}.$$

Proof : Let $G' = G \times H$. For the first inequality, given two minimal VD-colorings c_G and c_H of G and H respectively, color each edge $(a,x)(b,y)$ of G' with the color $(c_G(ab), c_H(xy))$. The color set of a vertex (a,x) of G' is then $S((a,x)) = S_{c_G}(a) \times S_{c_H}(x)$. Hence, for any two distinct vertices (a,x) and (b,y) of G', we have $S((a,x)) \neq S((b,y))$ since either $S_{c_G}(a) \neq S_{c_G}(b)$ or $S_{c_H}(x) \neq S_{c_H}(y)$, or both.

For the second inequality, without loss of generality, assume that $\min\{\chi'(G)\chi'_a(H), \chi'_a(G)\chi'(H)\} = \chi'_a(G)\chi'(H)$. Let c_G be an AVD-coloring of G with colors $0, 1, \ldots, \alpha-1$, where $\alpha = \chi'_a(G)$. Let γ_H be a proper edge coloring of H in $\chi'(H)$ colors.

An AVD-coloring c' of G' is obtained by setting for each $ab \in E(G),\ xy \in E(H)$:

$$c'((a,x)(b,y)) = (c_G(ab), \gamma_H(xy)).$$

The fact that c' is AVD is easily seen: we have $S_{c'}((a,x)) = S_{c_G}(a) \times S_{\gamma_H}(x)$. Hence, for any two adjacent vertices (a,x) and (b,y) of G', we have $S((a,x)) \neq S((b,y))$ since $S_{c_G}(a) \neq S_{c_G}(b)$ (a and b are adjacent vertices of G).

The third inequality can be shown in a same way, replacing χ'_a by χ''_a, AVD by TAVD and for each vertex (a,x) of G', setting $c'((a,x)) = (c_G(a), 0)$. □

Theorem 14 *For any graph G, the following holds*

$$\chi'_a(G \times K_2) \leq \chi'_a(G).$$

Moreover, equality holds if G is bipartite, or if G is regular and $\chi'_a(G) = \Delta(G)+1$, or if $\chi'_a(G) = \Delta(G)$.

Proof : An AVD-coloring c' of $G \times K_2$ is obtained by setting for $ab \in E(G),\ xy \in E(K_2)$:

$$c'((a,x)(b,y)) = c_G(ab).$$

In other words, we give the color of the edge ab to the edge $(a,x)(b,y)$. So, if $S_{c_G}(a) = \{a_1, a_2, \ldots, a_k\}$ then we have also $S_{c'}((a,x)) = \{a_1, a_2, \ldots, a_k\}$ and it is clear that c' is an AVD-coloring of $G \times K_2$. In the case where G is bipartite, then $G \times K_2$ consists in two disconnected copies of G and so $\chi'_a(G \times K_2) = \chi'_a(G)$. In the case where G is regular of degree Δ and $\chi'_a(G) = \Delta + 1$, then $G \times K_2$ is also regular and we have $\chi'_a(G \times K_2) \geq \Delta + 1$ and $\chi'_a(G \times K_2) \leq \chi'_a(G) = \Delta + 1$. So, $\chi'_a(G \times K_2) = \Delta + 1$. In the case where $\chi'_a(G) = \Delta$, then clearly, $\chi'_a(G \times K_2) = \Delta$.

□

Theorem 15 *For any $m \geq 3$, $n \geq 3$,*

$$\chi'_a(P_m \times P_n) = \begin{cases} 4 & \text{if } m = n = 3 \\ 5 & \text{otherwise.} \end{cases}$$

Proof : If $m = n = 3$, then $P_m \times P_n$ is the disjoint union of a 4-cycle and a 4-star. Hence four colors suffice to AVD-color $P_3 \times P_3$.

If $m \geq 3$, $n \geq 3$ and $m + n > 6$ then, in [BKT06], we proved that $\chi'_a(P_m \Box P_n) = 5$ and the coloring used is such that the four edges of each 4-length cycle in $P_m \Box P_n$ have pairwise different colors. Now, it is easily seen that $P_m \times P_n$ consists in two connected isomorphic components with two adjacent vertices of degree 4 (thus $\chi'_a(P_m \times P_n) \geq 5$) that are induced subgraphs of some grid $P_{m'} \Box P_{n'}$ (see Figure 2). Thus, two adjacent vertices of degree 4 in $P_m \times P_n$ have obviously different sets of colors. Moreover, two adjacent vertices of degree 2 have different sets of colors since they belong to a cycle of length 4 in $P_{m'} \Box P_{n'}$.

□

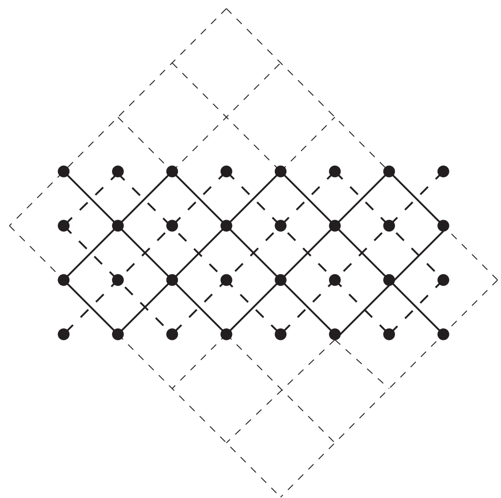

Figure 2: The product $P_8 \times P_4$ with a component (in bold) included in $P_6 \Box P_5$.

6.2 Strong product

Theorem 16 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \boxtimes H) \leq \min\{\chi'_s(G \Box H) + \chi'(G \times H), \chi'(G \Box H) + \chi'_s(G \times H)\},$$

$$\chi'_a(G \boxtimes H) \leq \chi'_a(G \Box H) + \chi'_a(G \times H),$$

$$\chi''_a(G \boxtimes H) \leq \chi'(G \boxtimes H) + \chi(G)\chi(H).$$

Proof : Remember that the edge set of $G \boxtimes H$ is the union of the edge set of $G \Box H$ and of $G \times H$. To obtain a VD-coloring of $G \boxtimes H$, VD-color the edges of $G \Box H$ in $\chi'_s(G \Box H)$ colors and properly color the edges of $G \times H$ in $\chi'(G \times H)$ new colors. Then the coloring is clearly proper and the vertices are distinguished by the VD-coloring of $G \Box H$. The same goes when exchanging the roles of G and H. We then obtain $\chi'_s(G \boxtimes H) \leq \min\{\chi'_s(G \Box H) + \chi'(G \times H), \chi'(G \Box H) + \chi'_s(G \times H)\}$.

For the AVD-coloring of $G \boxtimes H$, AVD-color the edges of $G \Box H$ in $\chi'_a(G \Box H)$ colors and AVD-color the edges of $G \times H$ in $\chi'_a(G \times H)$ new colors. This coloring is clearly proper and two adjacent vertices are distinguished either by the colors of the edges of $G \Box H$ or by the colors of the edges of $G \times H$.

The third inequality is a direct consequence of Claim 1 since $\chi(G \boxtimes H) \leq \chi(G)\chi(H)$ (see [IK00], page 246). □

6.3 Lexicographic product

Theorem 17 *For any two graphs G and H different from K_2, the following hold*

$$\chi'_s(G \circ H) \leq \chi'_s(G) + \chi'_s(H) + (|V(H)| - 1)\chi'(G),$$

$$\chi'_a(G \circ H) \leq \chi'_a(G) + \chi'_a(H) + (|V(H)| - 1)\chi'(G),$$

$$\chi''_a(G \circ H) \leq \chi'(G \circ H) + \chi(G)\chi(H).$$

Proof : Assume G is of order $n \geq 3$ and H is of order $m \geq 3$. The graph $G \circ H$ consists in n copies of H, two copies being linked by a complete bipartite graph $K_{m,m}$ if the corresponding vertices of G are adjacent. Thus, the edges between two copies of H can be decomposed into m perfect matchings. To obtain a VD-coloring of $G \circ H$:

- VD-color the edges of each copy of H with $\chi'_s(H)$ colors,
- for any edge e of G, color one of the m perfect matchings between the two copies of H corresponding to e with the color of e in a VD-coloring of G in $\chi'_s(G)$ new colors,
- properly color the edges of the remaining perfect matchings with $(m - 1)\chi'(G)$ new colors.

For the AVD-coloring, the proof is the same as above, replacing VD by AVD and χ'_s by χ'_a.

For the TAVD-coloring, as for the strong product, the third inequality follows from Claim 1 since $\chi(G \circ H) \leq \chi(G)\chi(H)$ ([IK00], page 246). □

7 Concluding remarks

We have obtained bounds for the VD, AVD and TAVD-chromatic indices of the Cartesian product of two graphs and we have shown that some of these bounds are optimal since they allow to determine the AVD and TAVD-chromatic indices of generalized hypercubes. General bounds for the direct, strong and lexicographic products have also been determined. Notice that, despite Theorems 3, 13, 16, 17 have quite simple proofs, they give the exact value of the adjacent vertex distinguishing chromatic index if the factors G and H are such that $\chi'_a(G) = \Delta(G)$ and $\chi'_a(H) = \Delta(H)$ (for instance, this is the case for trees with no two adjacent vertices of maximum degree [ZLW02]). However, the bounds we obtained for the TAVD-chromatic index of the direct, strong and lexicographic products do not seem to be very tight. It could be interesting to investigate this more in details.

References

[BBS02] P. N. Balister, B. Bollobás, and R. H. Schelp. Vertex distinguishing colorings of graphs with $\Delta(G) = 2$. *Discrete Math.*, 252(1-3):17–29, 2002.

[BGLS07] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp. Adjacent vertex distinguishing edge-colorings. *SIAM J. Discrete Math.*, 21(1):237–250 (electronic), 2007.

[BHBLW99] C. Bazgan, A. Harkat-Benhamdine, H. Li, and M. Woźniak. On the vertex-distinguishing proper edge-colorings of graphs. *J. Combin. Theory Ser. B*, 75(2):288–301, 1999.

[BHBLW01] C. Bazgan, A. Harkat-Benhamdine, H. Li, and M. Woźniak. A note on the vertex-distinguishing proper coloring of graphs with large minimum degree. *Discrete Math.*, 236(1-3):37–42, 2001. Graph theory (Kazimierz Dolny, 1997).

[BKLS04] P. N. Balister, A. Kostochka, Hao Li, and R. H. Schelp. Balanced edge colorings. *J. Combin. Theory Ser. B*, 90(1):3–20, 2004. Dedicated to Adrian Bondy and U. S. R. Murty.

[BKT06] J-L. Baril, H. Kheddouci, and O. Togni. Adjacent vertex distinguishing edge-colorings of meshes. *Australasian Journal of Combinatorics*, 35:89–102, 2006.

[BS97] A. C. Burris and R. H. Schelp. Vertex-distinguishing proper edge-colorings. *J. Graph Theory*, 26(2):73–82, 1997.

[ČHS96] J. Černý, M. Horňák, and R. Soták. Observability of a graph. *Math. Slovaca*, 46(1):21–31, 1996.

[EHW06] K. Edwards, M. Horňák, and M. Woźniak. On the neighbour-distinguishing index of a graph. *Graphs Combin.*, 22(3):341–350, 2006.

[GR06] C. Greenhill and A. Ruciński. Neighbour-distinguishing edge colorings of random regular graphs. *Electron. J. Combin.*, 13(1):Research Paper 77, 12 pp. (electronic), 2006.

[IK00] W. Imrich and S. Klavžar. *Product graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000. Structure and recognition, With a foreword by Peter Winkler.

[KM03] A. Kemnitz and M. Marangio. Total colorings of Cartesian products of graphs. In *Proceedings of the Thirty-Fourth Southeastern International Conference on Combinatorics, Graph Theory and Computing*, volume 165, pages 99–109, 2003.

[LWZW06] J. Li, W. Wang, Z. Zhang, and Z. Wang. Adjacent vertex-distinguishing total coloring on square, cube, biquadratic of pathes. *Int. J. Pure Appl. Math.*, 29(1):7–13, 2006.

[ZCL+05] Z. Zhang, X. Chen, J. Li, B. Yao, X. Lu, and J. Wang. On adjacent-vertex-distinguishing total coloring of graphs. *Sci. China Ser. A*, 48(3):289–299, 2005.

[ZCL+06] Z. Zhang, X. Chen, J. Li, B. Liang, and P. Qiu. A note on the relation of adjacent strong edge coloring and total coloring of graphs. Submitted, 2006.

[ZLW02] Z. Zhang, L. Liu, and J. Wang. Adjacent strong edge coloring of graphs. *Appl. Math. Lett.*, 15(5):623–626, 2002.

[ZŽ04] B. Zmazek and J. Žerovnik. Behzad-Vizing conjecture and Cartesian product graphs. In *Workshop on Graphs and Combinatorial Optimization*, volume 17 of *Electron. Notes Discrete Math.*, pages 297–300 (electronic). Elsevier, Amsterdam, 2004.

Appendix

$$c(K_6) = \left(\begin{array}{ccccccc c} & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \text{Missing colors} \\ x_0 & & 1 & 2 & 3 & 4 & 5 & 0,6 \\ x_1 & 1 & & 3 & 4 & 5 & 6 & 0,2 \\ x_2 & 2 & 3 & & 5 & 6 & 0 & 1,4 \\ x_3 & 3 & 4 & 5 & & 0 & 1 & 2,6 \\ x_4 & 4 & 5 & 6 & 0 & & 2 & 1,3 \\ x_5 & 5 & 6 & 0 & 1 & 2 & & 3,4 \end{array} \right)$$

$$c'(K_6') = \left(\begin{array}{ccccccc c} & x_0' & x_1' & x_2' & x_3' & x_4' & x_5' & \text{Missing colors} \\ x_0' & & 4 & 6 & 1 & 3 & 5 & 0,2 \\ x_1' & 4 & & 1 & 3 & 5 & 0 & 2,6 \\ x_2' & 6 & 1 & & 5 & 0 & 2 & 3,4 \\ x_3' & 1 & 3 & 5 & & 2 & 4 & 0,6 \\ x_4' & 3 & 5 & 0 & 2 & & 6 & 1,4 \\ x_5' & 5 & 0 & 2 & 4 & 6 & & 1,3 \end{array} \right)$$

$$d(K_8) = \left(\begin{array}{ccccccccc c} & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \text{Missing colors} \\ x_0 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0,8 \\ x_1 & 1 & & 3 & 4 & 5 & 6 & 7 & 8 & 0,2 \\ x_2 & 2 & 3 & & 5 & 6 & 7 & 8 & 0 & 1,4 \\ x_3 & 3 & 4 & 5 & & 7 & 8 & 0 & 1 & 2,6 \\ x_4 & 4 & 5 & 6 & 7 & & 0 & 1 & 2 & 3,8 \\ x_5 & 5 & 6 & 7 & 8 & 0 & & 2 & 4 & 1,3 \\ x_6 & 6 & 7 & 8 & 0 & 1 & 2 & & 3 & 4,5 \\ x_7 & 7 & 8 & 0 & 1 & 2 & 4 & 3 & & 5,6 \end{array} \right)$$

$$d'(K_8') = \left(\begin{array}{ccccccccc c} & x_0' & x_1' & x_2' & x_3' & x_4' & x_5' & x_6' & x_7' & \text{Missing colors} \\ x_0' & & 3 & 6 & 8 & 1 & 4 & 5 & 7 & 0,2 \\ x_1' & 3 & & 8 & 1 & 4 & 5 & 7 & 0 & 2,6 \\ x_2' & 6 & 8 & & 4 & 5 & 7 & 0 & 2 & 1,3 \\ x_3' & 8 & 1 & 4 & & 7 & 0 & 2 & 3 & 5,6 \\ x_4' & 1 & 4 & 5 & 7 & & 2 & 3 & 6 & 0,8 \\ x_5' & 4 & 5 & 7 & 0 & 2 & & 6 & 1 & 3,8 \\ x_6' & 5 & 7 & 0 & 2 & 3 & 6 & & 8 & 1,4 \\ x_7' & 7 & 0 & 2 & 3 & 6 & 1 & 8 & & 4,5 \end{array} \right)$$