The Irregularity Strength of Circulant Graphs (preliminary version)

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Abstract

The irregularity strength of a simple graph is the smallest integer k for which there exists a weighting of the edges with positive integers at most k such that all the weighted degrees of the vertices are distinct. In this paper we study the irregularity strength of circulant graphs of degree 4. We find the exact value of the strength for a large family of circulant graphs.

Keywords: Graph, Labeling, Irregularity Strength, Circulant graph.

1 Introduction and Definitions

All the graphs we deal with are undirected, simple and connected.

Let G = (V, E) be a graph with vertex set V and edge set E.

A function $w : E \to \mathbb{Z}^+$ is called a *weighting* of G, and for an edge $e \in E$, w(e) is called the *weight* of e. The *strength* s(w) of w is defined as $s(w) = \max_{e \in E} w(e)$. The *weighted degree* of a vertex $x \in V$ is the sum of the weights of its incident edges: $d_w(x) = \sum_{e \ni x} w(e)$. The *irregularity* strength s(G) of G is defined as $s(G) = \min\{s(w), w \text{ is an irregular weighting of } G\}$.

The study of s(G) was initiated by Chartrand et al. [CJL⁺88] and has proven to be difficult in general. There are not many graphs for which the irregularity strength is known. For an overview of the subject, the reader is referred to the survey of Lehel [Leh91] and recent papers [Jv95, JT95, AT98, ?, ?].

The irregularity strength of regular graphs was considered by several authors. For a regular graph G of order n and degree r, let $\lambda(G) = \lceil \frac{n+r-1}{r} \rceil$. A simple counting argument gives $s(G) \ge \lambda(G)$.

On the other hand, an upper bound of $\frac{n}{2} + 9$ was given for regular graphs of order n in [FL87]. This result was improved recently in [?]. Nevertheless, there is still a great gap between lower and upper bounds on the irregularity strength of regular graphs.

Moreover, for all the connected regular graphs for which the irregularity strength is known, we have $s(G) \leq \lambda(G) + 1$.

The following conjecture is due to Jacobson (see [Leh91, FL87]):

Conjecture 1 There exists an absolute constant c such that for each regular graph G, $s(G) \leq \lambda(G) + c$.

Circulant graphs are a large family of regular graphs which are studied in the context of interconnection networks [?] and have good properties such as symmetry, vertex-transitivity, ...

Definition 1 Let n be an integer and let s_1, s_2, \ldots, s_k be a sequence of integers, with $1 \le s_1 < s_2 < \ldots < s_k \le n/2$. The circulant graph $G = C_n(s_1, s_2, \ldots, s_k)$ of order n is a graph with vertex

set $V(G) = \{0, 1, ..., n-1\}$ and edge set $E(G) = \{(x, x \pm s_i \mod n), x \in V(G), 1 \le i \le k\}$. It is a Cayley graph on the additive group of integers modulo n, with set of generators $\{s_1, s_2, ..., s_k\}$.

In this paper, we study the irregularity strength of circulant graphs of degree 4. We find the exact value for the irregularity strength of circulant graphs of the form $C_n(1,k)$ with $n \ge 4k + 1$. These graphs consists in a cycle of length n plus chords joining vertices at distance k on the cycle.

We will use the following notations (see Figure 1): for any t, the vertices of $C_{4t+1}(1,k)$ are labeled y_0, y_1, \ldots, y_{2t} going clockwise on the cycle and $x_0 = y_0, x_1, x_2, \ldots, x_{2t}$ going counterclockwise.

For a weighting w on $C_{4t+1}(1,k)$, denote by a_i (resp. b_i) the weight of the edge (x_i, x_{i+1}) (resp. (y_i, y_{i+1})) for $0 \le i \le 2t - 1$, and by $a_{2t} = b_{2t}$ the weight of the edge (x_{2t}, y_{2t}) , and let



Figure 1: Notations

Let us give a construction of $C_{4(t+1)+1}(1,k)$ from $C_{4t+1}(1,k)$ useful for the proof of main results. Let $G = C_{4t+1}(1,k)$ and let $P_4 = (x_{2t+1}, x_{2t+2}, y_{2t+2}, y_{2t+1})$ be a path of length 3 where $t \ge k \ge 2$. We define the graph $G.P_4$ as follows :

$$\begin{split} V(G.P_4) &= V(G) \quad \bigcup V(P_4) \\ E(G.P_4) &= E(G) \quad \backslash (\bigcup_{i=1}^k \{(x_{2t-k+i}, y_{2t-i+1})\} \bigcup \{(x_{2t}, y_{2t})\}) \\ & \bigcup_{i=1}^k E(P_4) \\ & \bigcup_{i=1}^k \{(x_{2t-k+2+i}, y_{2t+3-i})\} \\ & \bigcup_{i=1}^k \{(x_{2t-k+1}, x_{2t+1}), (x_{2t-k+2}, x_{2t+2}), (y_{2t-k+1}, y_{2t+1}), (y_{2t-k+2}, y_{2t+2})\} \end{split}$$

Observe that $G.P_4 = C_{4t+5}(1,k)$. Thus for any $t \ge k \ge 2$, $C_{4t+1}(1,k)$ can be obtained by iterating (t-k) times this composition starting from $C_{4k+1}(1,k)$.

2 Preliminaries

Definition 2 For n = 4t + 1, with $t \ge k \ge 2$, $C_n(1,k)$ verifies the property \mathcal{P}_t if it admits an irregular weighting of strength $s = \lambda(C_n(1,k)) = t + 1$ such that :

i) $a_{2t} = b_{2t} = a_{2t-1} = b_{2t-1} + 1 = s$,

- *ii)* $c_{2t-i} = \bar{c}_{2t-i} = s$, for each $i, 0 \le i \le k$,
- *iii*) $a_{2t-2i+1} \ge b_{2t-2i+1} \ge \lceil \frac{k}{2} \rceil + 2 i$, for each $i, 0 \le i \le \lceil \frac{k}{2} \rceil 1$,
- *iv)* when k is even, $a_{2t-2k+2i+1} \ge i+1$, for each $i, 0 \le i \le k-1$.

Lemma 1 For $k \geq 2$, $C_{4k+1}(1,k)$ verifies the property \mathcal{P}_k .

Proof: In order to prove that $C_{4k+1}(1,k)$ verifies the property \mathcal{P}_k , we construct an irregular weighting w as follows:

$$\begin{array}{ll} a_0 = b_0 = 1 \\ a_i = \lfloor \frac{i+1}{2} \rfloor + 1 & \text{for each } i, \quad 1 \le i \le 2k \\ b_i = a_{i-1} & \text{for each } i, \quad 1 \le i \le 2k \\ c_i = \bar{c}_i = i+1 & \text{for each } i, \quad 0 \le i \le k-1 \\ c_i = \bar{c}_i = k+1 & \text{for each } i, \quad k \le i \le 2k \\ c_i = \bar{c}_i = 1 & \text{for each } i, \quad 1-k \le i \le -1 \end{array}$$

We can easily verify that we have $d_w(x_0) = d_w(y_0) = 4$, and for each $1 \le i \le 2k$:

$$\begin{cases} d_w(x_i) = a_i + a_{i-1} + c_i + c_{i-k} &= 4 + 2i \\ d_w(y_i) = b_i + b_{i-1} + c_i + c_{i-k} &= 3 + 2i \end{cases}$$

Consequently, w is irregular and of strength equal to $\lambda(C_{4k+1}(1,k)) = k+1$ (i.e, statements i) and ii) are verified). Moreover, by construction the remaining statements of property \mathcal{P}_k are also verified : the third statement is verified since $a_{2k-2i+1} \ge b_{2k-2i+1} = \lfloor \frac{2k-2i+1}{2} \rfloor + 1 \ge \lceil \frac{k}{2} \rceil - i + 2$. The fourth statement is also verified since $a_{2i+1} = \lfloor \frac{2i+2}{2} \rfloor + 1 = i+2 \ge i+1$. See part A of Appendix for an example when k = 4.

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3 Results

In this section, we present two propositions. The first one construct an irregular weighting for $C_{4t+1}(1,k)$ for any $t \ge k$. The second proposition allows to give an irregular weighting for the remaining graphs $C_n(1,k)$, with n = 4t + 2, 4t + 3, 4t + 4 for each $t \ge k$.

Proposition 1 For any t with $t \ge k \ge 2$, there exists an irregular weighting of $C_{4t+1}(1,k)$ which verifies the property \mathcal{P}_t .

Proof :

The proof is done by induction on t. For t = k, the result holds by Lemma 1.

Assume that there exists an irregular weighting of $C_{4t+1}(1,k)$, which verifies the property \mathcal{P}_t . We will construct a weighting of $C_{4t+5}(1,k)$ which verifies property \mathcal{P}_{t+1} .

Let w be an irregular weighting of $G = C_{4t+1}(1,k)$ which verifies the property \mathcal{P}_t . Let $G' = G.P_4 = C_{4t+5}(1,k)$ and w' be the weighting of G' constructed from the weighting w of G as follows:

$$w'(e) = \begin{cases} w(e) & \text{for } e \in E(G) \setminus M \\ w(e) - 1 & \text{for } e \in M \\ t + 2 & \text{for } e \in A \cup B \\ t + 1 & \text{for } e \in C \end{cases}$$

where,

$$M = \bigcup_{\substack{i=1\\k}}^{|\frac{\kappa}{2}|-1} \{ (x_{2t-2i+1}, x_{2t-2i+2}), (y_{2t-2i+1}, y_{2t-2i+2}) \}$$

$$A = \bigcup_{\substack{i=1\\k}}^{|k|} \{ (x_{2t-k+i}, y_{2t-i+3}) \}$$

$$B = \{ (x_{2t+2}, y_{2t+2}), (y_{2t-k+2}, y_{2t+2}), (x_{2t-k+2}, x_{2t+2}), (x_{2t+1}, x_{2t+2}) \}$$

$$C = \{ (x_{2t}, x_{2t+1}), (y_{2t}, y_{2t+1}), (x_{2t-k+1}, x_{2t+1}), (y_{2t+1}, y_{2t+2}), (y_{2t-k+1}, y_{2t+1}) \}$$

Note that, by statement *iii*) of property \mathcal{P}_t of G, $w(e) \ge 2$ if $e \in M$. Thus, $w'(e) \ge 1$ for each $e \in E(G')$. Moreover, the degrees of the vertices given by the weighting w' are as follows :

$$\begin{cases} d_{w'}(x_i) &= d_w(x_i) = 2i + 4 & \text{for } 0 \le i \le 2t - k + 1 \\ d_{w'}(y_i) &= d_w(y_i) = 2i + 3 & \text{for } 1 \le i \le 2t - k + 1 \end{cases}$$

Observe that for each of the following vertices, the weight of one adjacent edge has been decreased by one, and the weight of another adjacent edge has been increased by one. So the weighted degrees are unchanged : for $0 \le i \le \lfloor \frac{k}{2} \rfloor - 2$,

$$\begin{cases} d_{w'}(x_{2t-2i}) &= (a_{2t-2i-1}-1) + a_{2t-2i} + c_{2t-2i-k} + (c_{2t-2i}+1) = d_w(x_{2t-2i}) \\ d_{w'}(y_{2t-2i}) &= (b_{2t-2i-1}-1) + b_{2t-2i} + \bar{c}_{2t-2i-k} + (\bar{c}_{2t-2i}+1) = d_w(y_{2t-2i}) \\ d_{w'}(x_{2t-2i-1}) &= a_{2t-2i-1} + (a_{2t-2i}-1) + c_{2t-2i-k} + (c_{2t-2i}+1) = d_w(x_{2t-2i-1}) \\ d_{w'}(y_{2t-2i-1}) &= b_{2t-2i-1} + (b_{2t-2i}-1) + \bar{c}_{2t-2i-k} + (\bar{c}_{2t-2i}+1) = d_w(y_{2t-2i-1}) \\ \end{cases}$$

Moreover, when k is even, the weighted degrees of vertices x_{2t-k+2} and y_{2t-k+2} are not given above. Their weighted degrees are :

$$\begin{cases} d_{w'}(x_{2t-k+2}) &= a_{2t-k+1} + a_{2t-k+2} + c_{2t-2k+2} + c_{2t-k+1} = d_w(x_{2t-k+2}) + 1 \\ d_{w'}(y_{2t-k+2}) &= b_{2t-k+1} + b_{2t-k+2} + c_{2t-2k+2} + c_{2t-k+1} = d_w(y_{2t-k+2}) + 1 \end{cases}$$

We can see that, in this case, $d_{w'}(x_{2t-k+2}) = d_{w'}(y_{2t-k+3})$ and there does not exist a vertex $u \in V(G')$ such that $d_{w'}(u) = d_w(y_{2t-k+2})$ (see part B of Appendix).

For the vertices of P_4 we have :

$$\begin{cases} d_{w'}(x_{2t+1}) &= 4t+6\\ d_{w'}(x_{2t+2}) &= 4t+8\\ d_{w'}(y_{2t+1}) &= 4t+5\\ d_{w'}(y_{2t+2}) &= 4t+7 \end{cases}$$

Let a'_i, b'_i, c'_i and \bar{c}'_i be the values of edges of G' given by the weighting w' as defined in section 1. We distinguish two cases depending on the parity of k.

 $\underline{\text{Case 1}}: k \text{ is odd.}$

In order to verify the property \mathcal{P}_{t+1} for G', one can see that $1 \leq w'(e) \leq t+2$ for each $e \in E(G')$ and for each $u, v \in V(G')$, $d_{w'}(u) \neq d_{w'}(v)$ when $u \neq v$.

Thus w' is irregular and the statements i) and ii) hold. The third one iii) is also verified since

 $a'_{2(t+1)-2i+1} \ge b'_{2(t+1)-2i+1} = t+1 \ge \lceil \frac{k}{2} \rceil + 2 - i$ for i = 1 or 2, and

 $a_{2(t+1)-2i+1}^{2(t+1)-2i+1} \ge b_{2(t+1)-2i+1}^{2(t+1)-2i+1} \ge b_{2t-2(i-1)+1} - 1 \ge \lceil \frac{k}{2} \rceil + 2 - (i-1) - 1 \ge \lceil \frac{k}{2} \rceil + 2 - i \text{ for each } i, \\ 3 \le i \le \lceil \frac{k}{2} \rceil - 1.$

<u>Case 2</u>: k is even (a complete example is given in Appendix). We use a cycle C induced by the following vertices $C = (x_{2t-k+2}, x_{2t-k+1}, \ldots, x_{2t-2k+2})$. In order for the weighting to be irregular, it suffices to decrease by two the weighted degree of x_{2t-k+2} in w' (i.e, to obtain $d_{w'}(x_{2t-k+2}) = d_w(y_{2t-k+2})$) without changing degrees of other vertices.

To do this, we decrease and increase alternately by one the weight of the edges of C starting from x_{2t-k+2} . As C is odd, the two edges of C which are adjacent to x_{2t-k+2} are both decreased



Figure 2: An irregular weighting of C4t + 2(1, k) on the right is obtained from the irregular weighting of C4t + 1(1, k) on the left

by one. Thus the degree of x_{2t-k+2} is decreased by two. Moreover, for any other vertex of C, the degree is unchanged.

As G verified statement iv) of property \mathcal{P}_t , then the weights of edges of C which are decreased are greater or equal to one. In other hand, as the strength of G is equal to t + 1 and edges of C are in G, so by increasing some edges of C by one, the strength of w' remains $t + 2 = \lambda(G')$.

Observe that statements i), ii), iii) of property \mathcal{P}_{t+1} are proved similarly than in case k odd. So we now prove statement iv) for G'. Note that :

 $\begin{array}{l} a_{2(t+1)-2k+2i+1}' = a_{2t-2k+2(i+1)+1}' = a_{2t-2k+2(i+1)+1} - 1 \geq i+1 \text{ for } 1 \leq i \leq k-2. \text{ Moreover,} \\ \text{for } i = k-1, \, a_{2(t+1)-2k+2i+1}' = t+2 \geq k-1+1 = k. \end{array}$

The following proposition extends the irregular weighting to cases n = 4t + 2, 4t + 3 and 4t + 4, for $t \ge k$.

Proposition 2 If $C_{4t+1}(1,k)$ verifies property \mathcal{P}_t then there exists an irregular weighting of $C_{4t+1+i}(1,k)$, for any $i, 1 \leq i \leq 3$, of strength $s' = \lambda(C_{4t+1+i}(1,k)) = t+2$.

Proof: Observe that $\lambda(C_{4t+1+i}(1,k)) = \lceil \frac{4t+1+i+3}{4} \rceil = t+2$. We start from $G = C_{4t+1}(1,k)$ with an irregular weighting verifying property \mathcal{P}_t and we add *i* vertices, $1 \leq i \leq 3$, by subdividing the edge (x_{2t}, y_{2t}) in i+1 edges (see Figures 2, 3 and 4, where $\alpha_1, \alpha_2, \beta_1$ and β_2 are the weights of some edges of the irregular weighting of $C_{4t+1}(1,k)$).

The new edges and the edges whose endpoints have been modified are weighted as shown below:

Case i = 1: a weighting is obtained simply by subdividing the edge (x_{2t}, y_{2t}) in two and then by giving the weight t + 2 to the two created edges (see Figure 2).

Case i = 2: we distinguish between the case k odd and k even. The process for both cases is given in Figure 3, where dashed edges have weight t + 1 and bold edges are edges whose weights have been modified.

Case i = 3: we distinguish between the case k odd and k even. The process for both cases is given in Figure 4.

From proposition 1 and 2, we deduce the following result :

Theorem 1 For $n \ge 4k + 1$ and $k \ge 2$,

$$s(C_n(1,k)) = \left\lceil \frac{n+3}{4} \right\rceil.$$



Figure 3: An irregular weighting of $C_{4t+3}(1,k)$, with k odd (k = 5) on the left and k even (k = 4) on the right



Figure 4: An irregular weighting of $C_{4t+4}(1,k)$, with k odd (k = 5) on the left and k even (k = 4) on the right

4 Concluding remarks

We studied the irregularity strength of circulant graphs $C_n(1,k)$ when $k \leq \frac{n-1}{4}$ and found the exact value. Our method does not seem to work for $\frac{n}{4} \leq k \leq \frac{n}{2}$, because we have not found an irregular weighting to start the induction process.

But, the isomorphisms of circulant graphs allow us to come back to some cases when $k \leq \frac{n}{4}$. For instance, the following isomorphisms are easy to see :

$$\begin{cases} C_{2k-1}(1,k) \cong C_{2k-1}(1,2) \\ C_{3k-1}(1,k) \cong C_{3k-1}(1,3) \\ C_{4k-1}(1,k) \cong C_{4k-1}(1,4) \end{cases}$$

More generally, if gcd(n,k) = 1 then we have $C_n(1,k) \cong C_n(1,k^{-1} \mod n)$.

Moreover, our study is also generalized to some circulant graphs $C_n(s_1, s_2)$ with $s_1 \neq 1$ and $s_2 > s_1$. If $gcd(n, s_1) = 1$ or $gcd(n, s_2) = 1$ then $C_n(s_1, s_2) \cong C_n(1, k)$ for some k.

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Appendix : a complete example when k = 4

Let us give a complete example of the construction of an irregular weighting of $C_{21}(1,4)$ from $C_{17}(1,4)$. The values on the edges represent their weights and the bold values on vertices represent their weighted degrees.

Part A: an irregular weighting of $C_{17}(1,4)$ verifying property \mathcal{P}_4 .

Part B: the first step of the extension of the weighting of $C_{17}(1,4)$ to the graph $C_{21}(1,4)$ by composition with P_4 . Notice that the weighting is not irregular as there is two vertices of degree 17.

Part C: the weighting of part B is modified to become irregular by using a cycle of length 5 (in bold on the figure) to decrease the weighted degree of one of the two vertices of degree 17 to 15.

