Linear and cyclic radio k-labelings of Trees

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Abstract

Motivated by problems in radio channel assignments, we consider radio k-labelings of graphs. For a connected graph G and an integer $k \geq 1$, a linear radio k-labeling of G is an assignment f of non negative integers to the vertices of G such that

$$|f(x) - f(y)| \geq k + 1 - d_G(x, y),$$

for any two distinct vertices x and y, where $d_G(x, y)$ is the distance between x and y in G. A cyclic k-labeling of G is defined analogously by using the cyclic metric on the labels. In both cases, we are interested in minimizing the span of the labeling. The linear (cyclic, respectively) radio k-labeling number of G is the minimum span of a linear (cyclic, respectively) radio k-labeling of G.

In this paper, linear and cyclic radio k-labeling numbers of paths, stars and trees are studied. For the path P_n of order $n \leq k + 1$, we completely determine the cyclic and linear radio k-labeling numbers. For $1 \leq k \leq n - 2$, a new improved lower bound for the linear radio k-labeling number is presented. Moreover, we give the exact value of the linear radio k-labeling number of stars and we present an upper bound for the linear radio k-labeling number of trees.

Keywords: Graph theory; Radio channel assignment; Cyclic and linear radio k-labeling.

1 Introduction and Definitions

In wireless networks, an important task is the management of the radio spectrum, that is the assignment of radio frequencies to transmitters in a way that avoid interferences. Interferences can occur if transmitters with close locations receive close frequencies. The problem, often modeled as a coloring problem on the graph where vertices represent transmitters and edges indicate closeness of the transmitters, has been studied by several authors under different scenarios.

In this context, the general $L(p_1, p_2, \ldots, p_t)$-labeling problem has been proposed: find a labeling of the vertices of the graph such that the labels of any two vertices at distance d are p_d apart. The goal is to minimize the span (range of frequencies). Since this problem appears to be difficult in its generality, many particular cases have been studied. Among all, labelings with constraints at two distances, i-e. $L(h, k)$-labelings and particularly $L(2, 1)$-labeling introduced by Griggs and Yeh [7] have been the subject of many works.

In this paper, we study the radio k-labeling problem defined by Chartrand et al. [2, 4], which can be viewed as an extension of the $L(2,1)$-labeling problem and a particular case of the $L(p_1, p_2, \ldots, p_k)$-labeling problem in which the constraints are given by $p_i = k+1-i$. More formally, for a graph $G = (V, E)$, we denote by $d_G(x, y)$ the distance between two vertices x and y, and by $D(G)$ the diameter of G.

Definition 1 (linear and cyclic radio k-labeling)**.** *Let $G = (V, E)$ be a connected graph and k an integer with $k \geq 1$.*

i. *A linear radio k-labeling of G is a function f which maps each vertex of V to an element of $\{0, 1, 2, ..., \lambda^k(f)\}$ such that:*

$$|f(x) - f(y)| \geq k + 1 - d_G(x, y),$$

for every two distinct vertices x and y of G. $\lambda^k(f)$ is called the span of f. The linear radio k-labeling number $\lambda^k(G)$ of G is the minimum span of all linear radio k-labelings of G.

ii. *A cyclic radio k-labeling of G is a function g which maps each vertex of V to an element of $\{0, 1, 2, ..., \lambda_c^k(g) - 1\}$ such that for any $(x, y) \in V \times V$, $x \neq y$,*

$$k + 1 - d_G(x, y) \leq |g(x) - g(y)| \leq \lambda_c^k(g) - (k + 1 - d_G(x, y)),$$

or equivalently,

$$|g(x) - g(y)|_m \geq k + 1 - d_G(x, y),$$

where $|a - b|_m = \min\{|a - b|, m - |a - b|\}$ is the cyclic metric on the labels. ${\lambda_c}^k(g)$ is called the span of g. The cyclic radio k-labeling number ${\lambda_c}^k(G)$ of G is the minimum span of all cyclic radio k-labelings of G.

Let g be a cyclic k-labeling of G, we remark that if m is the largest label of g then the span of g is $m+1$ because there is one more interval of frequencies between m and 0.

Therefore, radio k-labelings generalize many known labelings: For $k = 1$, $\lambda^1(G) = \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. For $k = 2$, the radio 2-labeling problem corresponds to the $L(2,1)-$labeling problem. For $k = D(G) - 1$, a linear radio k-labeling is referred to as *radio antipodal coloring* (See [2, 1]), because only antipodal vertices can be of the same color. In that case, the minimum number of colors needed is called the *radio antipodal number*, denoted by $ac(G)$. As the authors of [2, 1] consider the maximum color used (labels are positive) instead of the span, then we have $\lambda^k(G) = ac(G) - 1$. Finally, in the case $k = D(G)$, $\lambda^k(G)$ is called the *radio number* and is studied in [3, 13].

Figure 1 shows a linear radio 3-labeling of a graph with span 10 (notice that the diameter of the graph is 4, thus this labeling is antipodal). If we consider this labeling to be a 13-labeling, then it is also a cyclic radio 3-labeling with span 13 (although the labels 11 and 12 are not used), but it is not a cyclic radio 3-labelings with span 12 since adjacent vertices with label 0 and 10 have too close labels.

In this paper, we shall relate the linear and cyclic radio k-labeling number with the following parameters: the *upper hamiltonian number* of a graph G of order n, denoted by $h^+(G)$ is the maximum of $\sum_{i=0}^{n-1} d_G(\pi(i+1), \pi(i))$, over all cyclic permutations π of the vertices of

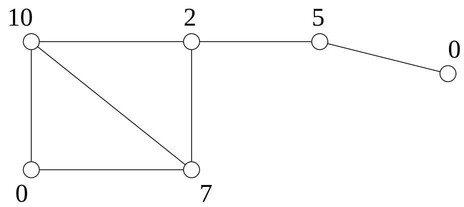

Figure 1: A linear radio 3-labelings with span 10.

G; and the *upper traceable number* (see [14]) is defined as $t^+(G) = \max_{\pi} \sum_{i=0}^{n-2} d_G(\pi(i+1), \pi(i))$. The upper hamiltonian number was first studied in [5, 6]. Later, Král et al. in [11] showed that the problem of determining the upper hamiltonian number of a graph is NP-hard. The same method can be used to prove that computing the upper traceable number is also an NP-hard problem.

The complexity of deciding whether $\lambda^k(G) \leq h$ for a given graph G and integers k and h is still unknown. However, in this paper, we show that determining the radio k-labeling number of a graph G for $k \geq 2D(G) - 2$ is an NP-hard problem.

To our knowledge, bounds on the linear radio k-labeling number are given only for the path P_n, when $k \leq n-1$. Furthermore, for the particular cases $k = n-2$ and $k = n-1$, the exact value of $\lambda^k(P_n)$ have been established recently [10, 13].

In the context of radio frequency assignment, many authors adopt the linear metric on the channel. Nevertheless, the use of the cyclic metric appears to be interesting also, even so not much works consider this approach: Heuvel et al. [8], Shepherd [15], and Leese and Noble [12] considered cyclic labeling, but mainly for constraints at two distances.

Notice that, although the authors in [4] only consider linear radio k-labelings for $k \leq D(G)$, one can also consider the case $k > D(G)$. The motivation behind the study of the case $k > D(G)$ is of two kinds: first, this case seems less difficult to study than the case $k \leq D(G)$ (see Theorem 1) and secondly, computing the radio k-labeling number of a graph for $k \geq D(G)$ can be help to compute the radio k-labeling number of other graphs with larger diameter, as it is done in [9].

In this paper, we first present a complete study both for the linear and the cyclic radio k-labeling problem on the path P_n for any value of n and k. In most cases we find the exact value of the radio numbers and in other cases we improved the existing bounds (see table below, where $(*)$ represents our results in this paper).

	$\lambda^k(P_n)$	$\lambda_c^k(P_n)$
$1 \le k \le n-3$	$\le \begin{cases} \frac{k^2+2k}{2} & \text{if } k \text{ is even,} \\ \frac{k^2+2k-1}{2} & \text{if } k \text{ is odd.} \end{cases}$ [4] $\ge \begin{cases} \frac{k^2+4}{2} & \text{if } k \text{ is even,} \\ \frac{k^2+1}{2} & \text{if } k \text{ is odd.} \end{cases}$ $(*)$	$\le \begin{cases} \frac{k^2+4k}{2} & \text{if } k \text{ is even,} \\ \frac{k^2+4k-1}{2} & \text{if } k \text{ is odd.} \end{cases}$ $(*)$ $\ge \begin{cases} \frac{k^2+2k+2}{2} & \text{if } k \text{ is even,} \\ \frac{(k+1)^2}{2} & \text{if } k \text{ is odd.} \end{cases}$ $(*)$
$k = n-2$	$= \begin{cases} 2p^2-4p+4 & \text{if } n = 2p, \\ 2p^2-2p+2 & \text{if } n = 2p+1. \end{cases}$ [10]	same bounds as above
$k = n-1$	$= \begin{cases} 2p^2-2p+1 & \text{if } n = 2p, \\ 2p^2+2 & \text{if } n = 2p+1. \end{cases}$ [13]	$= \begin{cases} 2p^2 & \text{if } n = 2p, \\ 2p^2+2p+1 & \text{if } n = 2p+1. \end{cases}$ $(*)$
$k \ge n$	$= \begin{cases} (n-1)k - \frac{1}{2}n(n-2) & \text{if } n \text{ is even,} \\ (n-1)k - \frac{1}{2}(n-1)^2+1 & \text{if } n \text{ is odd.} \end{cases}$ $(*)$	$= \begin{cases} nk - \frac{1}{2}n(n-2) & \text{if } n \text{ is even,} \\ nk - \frac{1}{2}(n-1)^2+1 & \text{if } n \text{ is odd.} \end{cases}$ $(*)$

Next, we investigate the linear and cyclic radio k-labeling number for trees. We give the exact value of the linear and cyclic radio k-labeling number of star for any $k \ge 2$. We also present an upper bound for the linear and cyclic radio k-labeling number of a tree of order at least 5 that is neither a star nor a path.

2 The linear radio k-labeling number of graphs

In this section, we first give some simple bounds on the linear radio k-labeling number of a general graph for any $k \ge 1$, and secondly, we present results concerning linear radio k-labeling numbers of paths.

2.1 General graphs

The following proposition helps to obtain an upper bound for $\lambda^k(G)$.

Proposition 1. ([10]) *For any graph G on n vertices, and for any integers $k \ge 1$ and $\alpha \ge 1$,*

$$\lambda^{k+\alpha}(G) \le \lambda^k(G) + (n-1)\alpha.$$

If G^k is the kth power of G, that is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{(x,y) \in V(G) \,|\, d_G(x,y) \le k\})$, then

Proposition 2. *For any integer $k \ge 1$,*

$$\lambda^k(G) \le k(\chi(G^k) - 1).$$

Proof. Assume that there exists a proper coloring f of G^k which uses $\chi(G^k)$ colors. Then, for any $(x,y) \in E(G^k)$, we have $|f(x) - f(y)| \ge 1$.

Consider the labeling $g : V(G) \to \{0, 1, ..., k.\chi(G^k) - k\}$, defined by $g(x) = kf(x)$ for any $x \in V(G)$. Let (x,y) be two distinct vertices of G. If $(x,y) \in E(G^k)$, we get

$$|g(x) - g(y)| = k|f(x) - f(y)| \ge k \ge k + 1 - d_G(x,y).$$

Hence, g is a linear k-labelings with span $\lambda^k(g) = k.\chi(G^k) - k$. Consequently,

$$\lambda^k(G) \le k(\chi(G^k) - 1).$$

□

Thus far we have presented only upper bounds on $\lambda^k(G)$ when $k \geq 1$. We now provide a lower bound on $\lambda^k(G)$. Note that the linear radio k-labeling number is related to the upper traceable number $t^+(G)$. This result also appears in our paper [9], but as the proof is short, we give it again to be complete.

Lemma 1. ([9]) *For any integer $k \geq 1$, and any graph G of order n,*

$$\lambda^k(G) \geq (n-1)(k+1) - t^+(G).$$

Proof. Let f be a linear radio k-labeling of G such that $\lambda^k(f) = \lambda^k(G)$. Let $(x_1, x_2, \ldots, x_n)$ be an ordering of the vertices of G such that $f(x_i) \leq f(x_{i+1})$. Observe that $f(x_n) - f(x_1) = \sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i))$.

As, for each $i = 1, \ldots, n-1$, $|f(x_{i+1}) - f(x_i)| \geq k + 1 - d_G(x_{i+1}, x_i)$, we get

$$f(x_n) - f(x_1) \geq (n-1)(k+1) - \sum_{i=1}^{n-1} d_G(x_{i+1}, x_i).$$

Thus,

$$\lambda^k(f) \geq (n-1)(k+1) - t^+(G).$$

$\square$

Finally, note that if $k \geq 2D(G) - 2$ then we can compute the exact value of $\lambda^k(G)$ as a function of k, n, and $h_o^+(G)$.

Theorem 1. ([9]) *For any graph G on n vertices, if $k \geq 2D(G) - 2$ then*

$$\lambda^k(G) = (n-1)(k+1) - t^+(G).$$

Remark 1. *An immediate consequence of this result is that given a graph G and an integer $k \geq 2D(G) - 2$, determining the radio k-labeling number of G is an NP-hard problem since computing the upper traceable number is NP-hard, as observed in the introduction.*

2.2 Paths

Let P_n denote the path on n vertices. In the next, we determine the exact value of the linear radio k-labeling number of P_n for $k \geq n$. Note that $\lambda^{n-2}(P_n)$ was calculated by Khennoufa and Togni in [10], and $\lambda^{n-1}(P_n)$ by Liu and Zhu in [13].

Theorem 2. ([10]) *For any $n \geq 5$,*

$$\lambda^{n-2}(P_n) = \begin{cases} 2p^2 - 4p + 4 & \textit{if } n = 2p, \\ 2p^2 - 2p + 2 & \textit{if } n = 2p + 1. \end{cases}$$

Theorem 3. ([13]) *For any $n \geq 3$,*

$$\lambda^{n-1}(P_n) = \begin{cases} 2p^2 - 2p + 1 & \textit{if } n = 2p, \\ 2p^2 + 2 & \textit{if } n = 2p + 1. \end{cases}$$

With these results in hand, we can now determine $\lambda^k(P_n)$ for $k \geq n$.

Lemma 2. ([13]) *For any integer $n \geq 2$,*

$$t^+(P_n) = \begin{cases} 2p^2 - 1, & \text{if } n = 2p. \\ 2p^2 + 2p - 1, & \text{if } n{=}2p{+}1; \end{cases}$$

Theorem 4. *If P_n is a path of order $n \geq 3$, then for $k \geq n$*

$$\lambda^k(P_n) = \begin{cases} (n-1)k - \frac{1}{2}n(n-2) & \text{if } n \text{ is even,} \\ (n-1)k - \frac{1}{2}(n-1)^2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $k \geq n$. For $n = 2p$, Proposition 1 gives

$$\lambda^k(P_n) \leq \lambda^{n-1}(P_n) + (n-1)(k-n+1).$$

With Theorem 3 we obtain

$$\lambda^k(P_n) \leq 2p^2 - 2p + 1 + (2p-1)(k-2p+1).$$

Then,

$$\lambda^k(P_n) \leq (n-1)k - \frac{1}{2}n(n-2).$$

On the other hand, by combining Lemma 1 and Lemma 2, we get

$$\lambda^k(P_n) \geq (n-1)k - \frac{1}{2}n(n-2).$$

For $n = 2p+1$, Proposition 1 gives

$$\lambda^k(P_n) \leq \lambda^n(P_n) + (n-1)(k-n).$$

In the next, we need to determine $\lambda^n(P_n)$. For this, we define a labeling f of P_n by

$$\begin{cases} f(i) = (2p+1)(i-1) + p + 1, & 1 \leq i \leq p+1, \\ f(i+p+1) = f(i) - p - 1, & 1 \leq i \leq p. \end{cases}$$

It is easy to verify that f is a linear radio k-labeling with span $\lambda^n(f) = f(p+1) = 2p^2+2p+1$. Then

$$\lambda^{2p+1}(P_{2p+1}) \leq 2p^2 + 2p + 1.$$

This implies

$$\lambda^k(P_n) \leq 2p^2 + 2p + 1 + (n-1)(k-n).$$

Thus

$$\lambda^k(P_n) \leq (n-1)k - \frac{1}{2}(n-1)^2 + 1.$$

On the other hand, by virtue of Lemma 1 and Lemma 2, we have

$$\lambda^k(P_n) \geq (n-1)k - \frac{1}{2}(n-1)^2 + 1.$$

□

Bounds for the radio k-labeling number of P_n where given in [4] :

Theorem 5. (Chartrand et al. [4]) *For* $1 \leq k \leq n-3$,

$$\lambda^k(P_n) \geq \begin{cases} \frac{k^2}{4}, & \text{if } k \text{ is even,} \\ \frac{k^2-1}{4}, & \text{if } k \text{ is odd.} \end{cases}$$

and

$$\lambda^k(P_n) \leq \begin{cases} \frac{k^2+2k}{2}, & \text{if } k \text{ is even,} \\ \frac{k^2+2k-1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

In the next, we give an improved lower bound for the linear radio k-labeling number of the path.

Theorem 6. *For* $1 \leq k \leq n-3$,

$$\lambda^k(P_n) \geq \begin{cases} \frac{k^2+4}{2}, & \text{if } k \text{ is even.} \\ \frac{k^2+1}{2}, & \text{if } k \text{ is odd,} \end{cases}$$

Proof. Observe that for any positive integers n and m, if $n \leq m$ then $\lambda^k(P_n) \leq \lambda^k(P_m)$. Thus, for $k \leq n-3$, we have $\lambda^k(P_{k+1}) \leq \lambda^k(P_n)$.
Moreover, according to Theorem 3, we have

$$\lambda^k(P_{k+1}) = \begin{cases} 2p^2 - 2p + 1, & \text{if } k+1 = 2p, \\ 2p^2 + 2, & \text{if } k+1 = 2p+1. \end{cases}$$

That is,

$$\lambda^k(P_{k+1}) = \begin{cases} \frac{k^2+1}{2}, & \text{if } k \text{ is odd,} \\ \frac{k^2+4}{2}, & \text{if } k \text{ is even.} \end{cases}$$

Consequently, for $k \leq n-3$, we have

$$\lambda^k(P_n) \geq \lambda^k(P_{k+1}) = \begin{cases} \frac{k^2+1}{2}, & \text{if } k \text{ is odd,} \\ \frac{k^2+4}{2}, & \text{if } k \text{ is even.} \end{cases}$$

□

3 The cyclic radio k-labeling number of graphs

In this section, we begin by presenting a relationship between $\lambda^k(G)$ and $\lambda_c^k(G)$ for a general graph G, and we provide lower and upper bounds for $\lambda_c^k(G)$. In the remainder we give a complete description of $\lambda_c^k(P_n)$, for all values of k and n.

3.1 General graphs

The following proposition is a consequence of more general result of Heuvel et al. in [8] concerning constraints $(p_1, p_2, ..., p_k)$. To be complete we give a short proof of this result.

Proposition 3. ([8]) *For any graph G and any integer $k \geq 1$,*

$$\lambda^k(G) + 1 \leq \lambda_c^k(G) \leq \lambda^k(G) + k.$$

Proof. By the definitions in Section 1, a cyclic radio k-labeling with span m is a linear radio k-labeling with span $m-1$. Thus, $\lambda^k(G) \leq \lambda_c^k(G) - 1$.
On the other hand, any linear radio k-labeling of G with span m is a cyclic radio k-labeling with span $m+k$ (note that by definition, it is not needed that the labeling is onto, and the span need not be used as a label on a vertex). Consequently, $\lambda_c^k(G) \leq \lambda^k(G) + k$. $\square$

Proposition 4. *For any graph G of order n and any integers $k \geq 1$ and $\alpha \geq 1$,*

$$\lambda_c^{k+\alpha}(G) \leq \lambda_c^k(G) + n\alpha.$$

Proof. Suppose that there exists a cyclic radio k-labelings f of G such that $\lambda_c^k(f) = \lambda_c^k(G)$. Let g be a labeling of G such that $g(x_i) = f(x_i) + i\alpha$ where $(x_1, x_2,...,x_n)$ is an ordering of V such that $f(x_i) \leq f(x_{i+1})$. So, for all $1 \leq i < j \leq n$, we have

$$\begin{aligned} |g(x_j) - g(x_i)| &= |f(x_j) - f(x_i)| + (j-i)\alpha \\ &\geq k + 1 - d_G(x_j, x_i) + (j-i)\alpha \\ &\geq k + \alpha + 1 - d_G(x_j, x_i) + (j-i-1)\alpha \\ &\geq k + \alpha + 1 - d_G(x_j, x_i) \end{aligned}$$

because $(j-i-1)\alpha \geq 0$. Moreover,

$$\begin{aligned} |g(x_j) - g(x_i)| &= |f(x_j) - f(x_i)| + (j-i)\alpha \\ &\leq \lambda_c^k(G) - (k + 1 - d_G(x_j, x_i)) + (j-i)\alpha \\ &\leq \lambda_c^k(G) + n\alpha - (k + \alpha + 1 - d_G(x_j, x_i)) - (n-1-(j-i))\alpha \\ &\leq \lambda_c^k(G) + n\alpha - (k + \alpha + 1 - d_G(x_j, x_i)) \end{aligned}$$

because $(n-1-(j-i))\alpha \geq 0$. Then, $\lambda_c^{k+\alpha}(g) = \lambda_c^k(G) + n\alpha$. Thus, $\lambda_c^{k+\alpha}(G) \leq \lambda_c^k(G) + n\alpha$. $\square$

The following result is an immediate consequence of propositions 2 and 3.

Proposition 5. *For any integer $k \geq 1$,*

$$\lambda_c^k(G) \leq k.\chi(G^k).$$

Like the linear radio k-labeling number which is related to the upper traceable number, the cyclic radio k-labeling number is related to the upper hamiltonian number $h^+(G)$ in this way:

Lemma 3. *Let G be a connected graph of order n, and let k be an integer. Then,*

$$\lambda_c^k(G) \geq n(k+1) - h^+(G).$$

Proof. Let f be a cyclic k-labeling labeling of G such that $\lambda_c^k(f) = \lambda_c^k(G)$. Let $(x_1, x_2,...,x_n)$ be an ordering of the vertices of G such that $f(x_i) \leq f(x_{i+1})$. Observe that

$$f(x_n) - f(x_1) = \sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i)).$$

As, for each $i = 1, ..., n-1$, $|f(x_{i+1}) - f(x_i)| \geq k + 1 - d_G(x_{i+1}, x_i)$, we get

$$f(x_n) - f(x_1) \geq (k+1)(n-1) - \sum_{i=1}^{n-1} d_G(x_{i+1}, x_i).$$

Now, as f is a cyclic labeling, we have

$$f(x_n) - f(x_1) \leq \lambda_c^k(f) - (k + 1 - d_G(x_n, x_1).$$

Combining the two inequalities, we obtain

$$\lambda_c^k(f) \geq n(k+1) - d_G(x_n, x_1) - \sum_{i=1}^{n-1} d_G(x_{i+1}, x_i).$$

As f is of minimal span, and $h^+(G) \geq d_G(x_n, x_1) + \sum_{i=1}^{n-1} d_G(x_{i+1}, x_i)$ by definition, then the lemma is proved. □

Theorem 7. *Let G be a graph on n vertices with diameter $D(G)$, then for any integers $k \geq 2D(G) - 2$,*

$$\lambda_c^k(G) = n(k+1) - h^+(G).$$

Proof. By Lemma 3 it suffices to prove

$$\lambda_c^k(G) \leq n(k+1) - h^+(G).$$

Combining Proposition 3 and Theorem 1, we obtain

$$\lambda_c^k(G) \leq (n-1)(k+1) - t^+(G) + k.$$

It is easy to verify that

$$h^+(G) \leq t^+(G) + d_G(\pi(n-1), \pi(0)) \leq t^+(G) + D(G).$$

where π is a permutation on the vertices of G.
As $2D(G) - 2 \leq k$, we get

$$\lambda_c^k(G) \leq n(k+1) - h^+(G).$$

□

3.2 Paths

The idea of the proof of the following result is the same as the one of Lemma 2 given in [13].

Lemma 4. *For any positive integers n,*

$$h^+(P_n) = \begin{cases} 2p^2, & \text{if } n = 2p, \\ 2p^2 + 2p, & \text{if } n=2p+1. \end{cases}$$

Proof. Let the vertex set of P_n be $\{0, 1, \ldots, n-1\}$. For a permutation π on the vertices, $d_G(\pi(i+1), \pi(i))$ is equal to either $\pi(i+1) - \pi(i)$ or $\pi(i) - \pi(i+1)$, whichever is positive. Therefore, each integer from $\{0, 1, \ldots, n-1\}$ occurs two times in the summation $\sum_{i=0}^{n-1} d_G(\pi(i+1), \pi(i))$, as positive or negative term, and in total we have n positive terms and n negative terms in the sum.

Thus, to maximize the sum, one has to take a permutation π such that the smaller numbers occur twice as negative and the greater numbers occur twice as positive.

If $n = 2p$ is even, then the configuration achieving the maximum summation is when each number in $\{0, 1, \dots, p-1\}$ occurs twice as negative and each of $\{p, p+1, \dots, 2p-1\}$ occurs twice as positive. In that case we obtain

$$\sum_{i=0}^{2p-1} d_G(\pi(i+1), \pi(i)) = 2(\sum_{i=p}^{2p-1} i - \sum_{i=0}^{p-1} i) = 2\sum_{i=0}^{p-1} p = 2p^2.$$

If $n = 2p+1$ is odd, then the configuration achieving the maximum summation is when each number in $\{0, 1, \dots, p-1\}$ occurs twice as negative, each of $\{p+1, p+2, \dots, 2p\}$ occurs twice as positive and p occurs once as negative and once as positive. In that case we obtain

$$\sum_{i=0}^{2p-1} d_G(\pi(i+1), \pi(i)) = 2(\sum_{i=p+1}^{2p} i - \sum_{i=0}^{p-1} i) - p + p = 2\sum_{i=0}^{p-1}(p+1) = 2p(p+1).$$

□

Using this result, we can compute the cyclic radio k-labeling number of the path P_n for $k = n-1$.

Theorem 8. *For any $n \geq 3$,*

$$\lambda_c^{n-1}(P_n) = \begin{cases} 2p^2, & \text{if } n = 2p, \\ 2p^2 + 2p + 1, & \text{if } n = 2p+1. \end{cases}$$

Proof. If $k = n-1$, then according to Proposition 3 and Theorem 3, we get

$$\lambda_c^{2p-1}(P_{2p}) \leq 2p^2.$$

Therefore, it suffices to show that $\lambda_c^{2p-1}(P_{2p}) \geq 2p^2$. Lemma 3 and Lemma 4 give

$$\lambda_c^{2p-1}(P_{2p}) \geq 4p^2 - 2p^2 = 2p^2.$$

For $n = 2p+1$, we define a labeling f of P_{2p+1} by

$$\begin{cases} f(i) = (2p+1)i, & 1 \leq i \leq p. \\ f(p+1) = p, & \\ f(i+p+1) = f(i) + p, & 1 \leq i \leq p. \end{cases}$$

Note that the span of f is $\lambda_c^{2p}(f) = f(2p+1) + 1 = 2p^2 + 2p + 1$.

Now, we need to verify that f is a cyclic radio $(2p)$-labeling. For $1 \leq i, j \leq 2p+1$, let

- $A_{i,j} = |f(i) - f(j)|$,
- $\Delta_{i,j} = k + 1 - d_{P_n}(i,j) = 2p + 1 - |i-j|$,
- $\overline{\Delta}_{i,j} = \lambda_c^{2p-1}(f) - \Delta_{i,j} = 2p^2 + |i-j|$.

Case 1: if $1 \le i < j \le p$, then

- $A_{i,j} - \Delta_{i,j} = (2p+1)(j-i) - (2p+1) + (j-i) = (2p+2)(j-i-1) + 1 \ge 0$.
  $\overline{\Delta}_{i,j} - A_{i,j} = 2p^2 + (j-i) - (2p+1)(j-i) = 2p(p-(j-i)) \ge 0$. Thus,
$$\Delta_{i,j} \le A_{i,j} \le \overline{\Delta}_{i,j}.$$
- Observe that $\Delta_{i+p+1,j+p+1} = \Delta_{i,j}$, and $A_{i+p+1,j+p+1} = A_{i,j}$, and $\overline{\Delta}_{i+p+1,j+p+1} = \overline{\Delta}_{i,j}$, then
$$\Delta_{i+p+1,j+p+1} \le A_{i+p+1,j+p+1} \le \overline{\Delta}_{i+p+1,j+p+1}.$$
- $A_{i,j+p+1} - \Delta_{i,j+p+1} = (2p+1)(j-i) + p - (p-(j-i)) = (2p+2)(j-i) \ge 0$.
  and $\overline{\Delta}_{i,j+p+1} - A_{i,j+p+1} = 2p^2+p+1+(j-i)-((2p+1)(j-i)+p) = 2p(p-(j-i))+1 \ge 0$. Thus
$$\Delta_{i,j+p+1} \le A_{i,j+p+1} \le \overline{\Delta}_{i,j+p+1}.$$
- $A_{i+p+1,j} - \Delta_{i+p+1,j} = (2p+1)(j-i) - p - (p+(j-i)) = 2p(j-i-1) \ge 0$.
  and $\overline{\Delta}_{i+p+1,j} - A_{i+p+1,j} = 2p^2+p+1-(j-i)-((2p+1)(j-i)-p) = (2p+2)(p-(j-i))+1 \ge 0$. Thus
$$\Delta_{i+p+1,j} \le A_{i+p+1,j} \le \overline{\Delta}_{i+p+1,j}.$$

Case 2: if $1 \le i \le p$, then

- $A_{i,p+1} - \Delta_{i,p+1} = (2p+1)i - p - (p+i) = 2p(i-1) \ge 0$.
  $\overline{\Delta}_{i,p+1} - A_{i,p+1} = (2p+2)(p-i) + 1 \ge 0$. Thus,
$$\Delta_{i,p+1} \le A_{i,p+1} \le \overline{\Delta}_{i,p+1}.$$
- $A_{i+p+1,p+1} - \Delta_{i+p+1,p+1} = (2p+1)i - (2p+1-i) = (2p+2)(i-1) + 1 \ge 0$.
  $\overline{\Delta}_{i+p+1,p+1} - A_{i+p+1,p+1} = 2p(p-i) \ge 0$. Thus,
$$\Delta_{i+p+1,p+1} \le A_{i+p+1,p+1} \le \overline{\Delta}_{i+p+1,p+1}.$$

Thus, f is a cyclic radio $2p$-labeling of P_{2p+1}. Hence,
$$\lambda_c^{2p}(P_{2p+1}) \le \lambda_c^{2p}(f) = 2p^2 + 2p + 1.$$
By Lemma 3 and Lemma 4, we obtain
$$\lambda_c^{2p}(P_{2p+1}) = 2p^2 + 2p + 1.$$
□

Theorem 9. *For any $k \ge n \ge 3$*
$$\lambda_c{}^k(P_n) = \begin{cases} nk - \frac{1}{2}n(n-2) & \text{if } n \text{ is even,} \\ nk - \frac{1}{2}(n-1)^2 + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The result follows by combining Theorem 4 and Proposition 3. □

Theorem 10. *For $1 \leq k \leq n-2$,*

$$\lambda_c^k(P_n) \leq \begin{cases} \frac{k^2+4k}{2}, & \text{if } k \text{ is even.} \\ \frac{k^2+4k-1}{2}, & \text{if } k \text{ is odd;} \end{cases}$$

Proof. The result follows by combining Theorem 5 and Proposition 3. $\square$

Theorem 11. *For $1 \leq k \leq n-2$,*

$$\lambda_c^k(P_n) \geq \begin{cases} \frac{k^2+2k+2}{2}, & \text{if } k \text{ is even.} \\ \frac{(k+1)^2}{2}, & \text{if } k \text{ is odd;} \end{cases}$$

Proof. As for the linear case, we can show that for any positive integers n and m, if $n \leq m$ then $\lambda_c^k(P_n) \leq \lambda_c^k(P_m)$. This allows us to prove the theorem. $\square$

4 Linear and cyclic radio k-labeling of trees

In this section, we first determine the exact value of the linear and cyclic radio k-labeling number of stars for any $k \geq 2$. Secondly, we give the upper bound of the linear and cyclic radio k-labeling number of a tree of order at least 5 that is neither a star nor a path.

4.1 Linear and cyclic radio k-labeling of stars

Let S_n be a star on n vertices.

Theorem 12. *For any integer $n \geq 3$ and for any integer $k \geq 2$,*

$$\lambda^k(S_n) = (n-1)(k-1)+1,$$

and

$$\lambda_c^k(S_n) = n(k-1)+2.$$

Proof. For any integer $n \geq 3$, we have $D(S_n) = 2$. Since $k \geq 2D(S_n) - 2$, by Theorem 1, we get

$$\lambda^k(S_n) = (n-1)(k+1) - t^+(S_n).$$

Let $\{0, 1, \cdots, n-1\}$ be the vertex set of S_n. It is easy to see that

$$t^+(S_n) = \max_{\pi} \sum_{i=0}^{n-2} d_{S_n}(\pi(i+1), \pi(i)) = 1 + \sum_{i=1}^{n-2} 2 = 2n-3.$$

This implies

$$\lambda^k(S_n) = (n-1)(k-1)+1.$$

In other hand, Proposition 3 and Lemma 3 give

$$n(k+1) - h^+(S_n) \leq \lambda_c^k(S_n) \leq \lambda^k(S_n) + k.$$

Easy calculation shows that $h^+(S_n) = 2n-2$. Then

$$n(k-1)+2 = n(k+1) - 2n + 2 \leq \lambda_c^k(S_n) \leq (n-1)(k-1)+1+k = n(k-1)+2.$$

Thus

$$\lambda_c^k(S_n) = n(k-1)+2.$$

$\square$

4.2 Linear and cyclic radio k-labeling of trees

Let T_n be a tree of order $n \geq 5$ that is neither a star nor a path.

Lemma 5. *There exists a Hamiltonian path $(x_0, x_1, \cdots, x_{n-1})$ in the complement $\overline{T}_n$ of T_n and there exists $0 \leq p \leq n-2$ such that $d_{T_n}(x_p, x_{p+1}) \geq 3$.*

Proof. For proof, we proceed by induction on the order of T_n.

For $n = 5$, by symmetric reason, two possible configurations of T_5 are presented in Figure 2. In both case the path $(x_0, x_1, x_2, x_3, x_4)$ is a Hamiltonian path in $\overline{T}_5$ such that $d_{T_5}(x_p, x_{p+1}) = 3$ with $p = 1$ or 2.

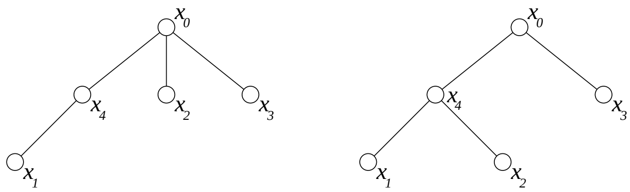

Figure 2: The two possible configurations for T_5.

Suppose that the result holds for every tree T_n of order $n \geq 5$. It is easy to see that T_{n+1} contains a leaf x such that $T_{n+1} - x$ is a tree that is neither a star nor a path. Thus, by the induction hypothesis, there exists a Hamiltonian path $(x_0, x_1, \cdots, x_{n-1})$ in the complement $\overline{T_{n+1} - x}$ of $T_n - x$, and there exists $0 \leq p \leq n-2$ such that $d_{T_n}(x_p, x_{p+1}) \geq 3$. Since x is a leaf in T_{n+1}, x is adjacent to one vertex of $(x_0, x_1, \cdots, x_{n-1})$. If x is adjacent to x_0 in T_{n+1} then x is not adjacent to x_{n-1} in T_{n+1}, so $(x_0, x_1, \cdots, x_{n-1}, x)$ is a Hamiltonian path in $\overline{T}_{n+1}$. If x is adjacent to x_{n-1} in T_{n+1} then x is not adjacent to x_0 in T_{n+1}, thus $(x, x_0, x_1, \cdots, x_{n-1})$ is a Hamiltonian path in $\overline{T}_{n+1}$. Thus, either $(x_0, x_1, \cdots, x_{n-1}, x)$ or $(x, x_0, x_1, \cdots, x_{n-1})$ is a Hamiltonian path in $\overline{T}_{n+1}$. Moreover, in both case we have $d_{T_{n+1}}(x_p, x_{p+1}) \geq 3$. $\square$

Theorem 13. *For any integer $k \geq 2$,*

$$\lambda^k(T_n) \leq (n-1)(k-1) - 1.$$

Proof. As T_n is a tree of order $n \geq 5$ that is neither a star nor a path, by Lemma 5, $\overline{T}_n$ contains a Hamiltonian path $(x_0, x_1, \cdots, x_{n-1})$ and there exists p with $0 \leq p \leq n-2$ such that $d_{T_n}(x_p, x_{p+1}) \geq 3$.

Since $(x_0, x_1, \cdots, x_{n-1})$ is a Hamiltonian path in $\overline{T}_n$, it follows that for each $0 \leq i \leq n-2$, $d_{T_n}(x_i, x_{i+1}) \geq 2$. Define a labeling f of T_n by

$$f(x_i) = \begin{cases} (i-1)(k-1), & \text{if } 0 \leq i \leq p, \\ (i-1)(k-1) - 1, & \text{if } p+1 \leq i \leq n-2. \end{cases}$$

If $0 \leq i < j \leq p$ or $p+1 \leq i < j \leq n-2$ then $|f(x_j) - f(x_i)| = (j-i)(k-1)$.

If $j - i = 1$ then $d_{T_n}(x_i, x_{i+1}) \geq 2$, and we get $|f(x_{i+1}) - f(x_i)| = k - 1 \geq k + 1 - d_{T_n}(x_{i+1}, x_i)$. Moreover, if $j - i \geq 2$ we can easily verify that $|f(x_j) - f(x_i)| = 2(k-1) \geq k + 1 - d_{T_n}(x_j, x_i)$.

If $0 \leq i \leq p$ and $p+1 \leq j \leq n-2$ then

$$|f(x_j) - f(x_i)| = (j-i)(k-1) - 1.$$

If $i = p$ and $j = p+1$ then $|f(x_{p+1}) - f(x_p)| = k-2$. Since $d_{T_n}(x_{p+1}, x_p) \geq 3$, we get $|f(x_{p+1}) - f(x_p)| \geq k+1-d_{T_n}(x_{p+1}, x_p)$.

If $j-i \geq 2$ then $|f(x_j)-f(x_i)| \geq 2k-3$, in this case it is easy to verify that $|f(x_j)-f(x_i)| \geq k+1-d_{T_n}(x_j, x_i)$.

Thus f is a linear radio k-labeling of T_n of span $\lambda^k(f) = f(x_{n-1}) = (n-1)(k-1)-1$. Consequently,

$$\lambda^k(T_n) \leq \lambda^k(f) \leq (n-1)(k-1)-1.$$

□

The corollary follows by combining Theorem 13 and Proposition 3

Corollary 1. *For any integer $k \geq 2$,*

$$\lambda_c^k(T_n) \leq n(k-1).$$

5 Concluding remarks

For paths P_n, we have presented exact results for the linear radio k-labeling number for some values of k and quite close upper and lower bounds. Nevertheless, it seems to be difficult to find an exact formula for $\lambda^k(P_n)$ for any k.

Several examples lead us to conjecture that the upper bound of Chartrand et al. is the exact value when n is large enough, i-e. that

$$\lim_{n\to+\infty} \lambda^k(P_n) = \begin{cases} \frac{k^2+2k}{2}, & \text{if } k \text{ is even,} \\ \frac{k^2+2k-1}{2}, & \text{if } k \text{ is odd.} \end{cases}$$

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