The Radio Antipodal and Radio Numbers of the Hypercube

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Abstract

A radio k-labeling of a connected graph G is an assignment f of non negative integers to the vertices of G such that

$$|f(x) - f(y)| \geq k + 1 - d(x, y),$$

for any two vertices x and y, where $d(x, y)$ is the distance between x and y in G. The radio antipodal number is the minimum span of a radio $(\mathrm{diam}(G) - 1)$-labeling of G and the radio number is the minimum span of a radio $(\mathrm{diam}(G))$-labeling of G.

In this paper, the radio antipodal number and the radio number of the hypercube are determined by using a generalization of binary Gray codes.

Keywords: graph labeling, radio antipodal number, radio number, generalized binary Gray code.

1 Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. The distance between two vertices u and v of G is denoted by $d(u, v)$ and the diameter of G is denoted by $\mathrm{diam}(G)$. A *radio k-labeling* f of G is an assignment of non negative integers to the vertices of G such that

$$|f(u) - f(v)| \geq k + 1 - d(u, v),$$

for every two distinct vertices u and v of G. The span of the function f denoted by $\mathrm{rc}_k(f)$, is $\max\{f(x) - f(y) : x, y \in V(G)\}$. The *radio k-chromatic number* $\mathrm{rc}_k(G)$ of G is the minimum span of all radio k-labelings of G.

Radio k-labelings were introduced by Chartrand et al. [1], motivated by radio channel assignment problems with interference constraints. Quite few results are known concerning radio k-labelings. The radio k-chromatic number for paths was studied in [1], where lower and upper bounds were given. These bounds have been improved in [8]. Radio k-labelings were also studied in relation with the Cartesian product of graphs [7].

Radio k-labelings generalize many other graph labelings. A radio 1-labeling is a proper vertex-colouring and $\mathrm{rc}_1(G) = \chi(G) - 1$. For $k = 2$, the radio 2-labeling problem corresponds to the well studied $L(2,1)$-labeling problem (see for instance [6, 10] and references therein). Large values of k (close to the diameter of the graph) were also considered for radio k-labelings.

For $k = \mathrm{diam}(G) - 1$, a radio k-labeling is referred to as a *(radio) antipodal labeling*, because only antipodal vertices can have the same label. The minimum span of an antipodal labeling is called the *(radio) antipodal number*, denoted by $\mathrm{an}(G)$. Radio antipodal labelings were first studied by Chartrand et al. in [1, 2] where bounds for the antipodal number for path and cycles were given. In [3] Chartrand et al., gave general bounds for the antipodal number of a graph. In [9], the authors determined the exact value of the radio antipodal number of paths. Recently, Justie and Liu have almost completely determined the radio antipodal number of the cycle [12].

A radio k-labeling with $k = \mathrm{diam}(G)$ is known as a *radio labeling* (or multi-level distance labeling in [14]). The minimum span of a radio labeling is called the *radio number*, denoted by $\mathrm{rn}(G)$. For paths and cycles, the radio number was studied by Chartrand et al. [4] and by Zhang [16] and completely determined by Liu and Zhu [14]. The radio number for square of paths and of cycles was investigated by Liu and Xie [11, 13]. More recently, Liu have studied the radio number of trees [15].

Notice that the authors of [1, 2, 3, 4] assume that the labels (colours) are positive. However, when speaking about labelings in relation with frequency assignment, it is more common to use non negative integers as labels. Thus the notation of the present paper follows the terminology of [14, 11, 13, 12] in which vertices are labelled by non negative integers.

The hypercube Q_n of dimension n has binary n-bit strings as vertex set, two vertices being adjacent if the corresponding strings differ on exactly one position.

Upper and lower bounds for the radio k-chromatic number of the hypercube Q_n were given in [7].

Theorem 1 ([7]). *For the hypercube Q_n of dimension $n \geq 2$ and for any $k \geq 2$,*

$$(2^n - 1)k - 2^{n-1}(2n - 3) + n - 2 \leq \mathrm{rc}_k(Q_n) \leq (2^n - 1)k - 2^{n-1} + 1.$$

Moreover, for $k \geq 2n - 2$,

$$\mathrm{rc}_k(Q_n) = (2^n - 1)k - 2^{n-1} + 1.$$

However, these bounds are quite far from being optimal, specially when k is close to the diameter of Q_n.

The aim of this paper is to determine the radio antipodal number and the radio number of the hypercube Q_n by showing:

Theorem 2. *For any positive integer $n \geq 1$,*

$$\mathrm{an}(Q_n) = (2^{n-1} - 1)\lceil \frac{n}{2} \rceil + \varepsilon(n),$$

with $\varepsilon(n) = 1$ if $n \equiv 0 \bmod 4$, $\varepsilon(n) = 0$ otherwise.

Theorem 3. *For any positive integer $n \geq 1$,*

$$\mathrm{rn}(Q_n) = (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1.$$

Example of minimal antipodal and radio labelings of Q_4 and Q_5 are given in Figure 1, showing that $\mathrm{an}(Q_4) = 15$, $\mathrm{an}(Q_5) = 45$, $\mathrm{rn}(Q_4) = 29$ and $\mathrm{rn}(Q_5) = 61$.

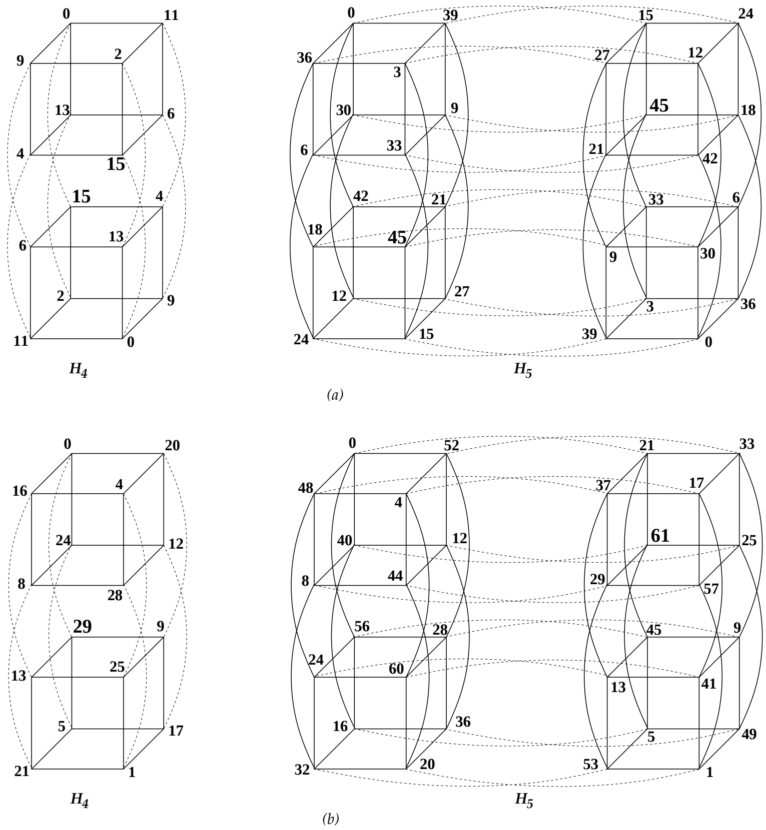

Figure 1: Optimal antipodal labelings (part *(a)* on the top) and radio labelings (part *(b)* on the bottom) for Q_4 and Q_5.

The proofs of these theorems mainly rely on finding an ordering of the vertices of Q_{n-1} with some prescribed distance (approximately half the diameter) between successive vertices. This ordering is presented in Section 2 as a generalized binary Gray code since we find more convenient to use the Gray code terminology. With this ordering in hand, we shall construct in Section 3 an antipodal labeling and a radio labeling of the hypercube and we will show that they are both optimal.

2 Generalized binary Gray codes

Define $\overline{0} = 1$ and $\overline{1} = 0$.

Notation 1. *For any two binary strings* $a = a_0a_1\cdots a_{n-1}$ *and* $b = b_0b_1\cdots b_{n'-1}$, *let*

- $|a| = n$,
- $\overline{a} = \overline{a_0}\ \overline{a_1}\cdots\overline{a_{n-1}}$,
- $I(a) = \overline{a_0}\ a_1\cdots a_{n-1}$,
- $a \uplus b = a_0a_1\cdots a_{n-1}b_0b_1\cdots b_{n'-1}$.

The *Hamming distance* $d_H(a, b)$ between two n-bit strings a and b is the number of bits in which they differ.

A *binary Gray code* is an ordered list of all the binary strings of length n such that successive items have Hamming distance exactly one.

A Gray code therefore can be viewed as a Hamiltonian path (or cycle if the code is circular, i.e. if the distance between the first and the last item of the code is one) in the hypercube Q_n.

There are many binary Gray codes. in the rest of this paper, we shall use the *binary reflected Gray code* $\mathcal{B}_n$ [5] which can be constructed recursively : $\mathcal{B}_1 = (0, 1)$ and to obtain $\mathcal{B}_n$, list $\mathcal{B}_{n-1}$ and next to it list $\mathcal{B}_{n-1}$ in reverse; then prepend 0s to the first half and 1s to the second half. Table 1 illustrates the structure of this code.

00. . . 0000
00. . . 0001
00. . . 0011
00. . . 0010
00. . . 0110
⋮
01. . . 0001
01. . . 0000
11. . . 0000
11. . . 0001
⋮
10. . . 0110
10. . . 0010
10. . . 0011
10. . . 0001
10. . . 0000

Table 1: Structure of the Gray code $\mathcal{B}_n$

The elements of a list $\mathcal{L}$ of q items, will be numbered from 0 to $q-1$ and we shall denote by $\mathcal{L}(i)$ the element number i, $0 \le i \le q-1$.

Now, we shall introduce two variation of Gray codes that we will use in the next section.

Definition 1. *A binary* (n, ℓ)*-Gray code (quasi* (n, ℓ)*- Gray code respectively) is a listing of all the* n*-bit strings such that the Hamming distance between two* *successive strings is exactly ℓ (ℓ, except between the two items $2^{n-1}-1$ and 2^{n-1} for which it is $\ell-1$ or $\ell+1$, respectively).*

Remark that a binary $(n,1)$-Gray code is a binary Gray code. An example of a $(5,3)$-Gray code is given in Figure 2.

Remark 1. *Notice that for even ℓ, a (n,ℓ)-Gray code cannot exist for parity reason: if the list starts with a string containing an even number of 1s, as an even number (ℓ) of bits are changed between successive items, all the strings of the code will have an even number of 1s. Thus we miss half of all the 2^n strings. This is the reason why we define quasi (n,ℓ)-Gray codes.*

Lemma 1. *Let n and ℓ be two positive integers. If $n>\ell$ then there exists a quasi (n,ℓ)-Gray code.*

Proof. Let n,m and ℓ be three positive integers with $n>\ell$ and $m=n-\ell+1$.

Define the lists $\mathcal{L}_i$ constructed by using two binary reflected Gray codes in this way: for all $0\le i\le 2^{\ell-2}-1$ and $0\le j\le 2^m-1$,

$$\mathcal{L}_i(j)=\begin{cases}\mathcal{B}_m(j)\uplus\mathcal{B}_{\ell-1}(i) & \text{if } j \text{ is even},\\ \mathcal{B}_m(j)\uplus\overline{\mathcal{B}_{\ell-1}(i)} & \text{if } j \text{ is odd}.\end{cases}$$

Thus, the Hamming distance between two successive items in each list $\mathcal{L}_i$ is ℓ and $|\mathcal{L}_i(j)|=n$.

We define the lists $\mathcal{L}_i^{'}$ by $\mathcal{L}_i^{'}(j)=I(\mathcal{L}_i(j))$, for $0\le j\le 2^m-1$ and $0\le i\le 2^{\ell-2}-1$.

From the lists $\mathcal{L}_i$ and $\mathcal{L}_i^{'}$, we construct two lists $\mathcal{G}^1_{n,\ell}$ and $\mathcal{G}^2_{n,\ell}$ (which are partial (n,ℓ)-Gray codes) in this way: for $0\le i\le 2^{\ell-2}-1$ and $0\le j\le 2^m-1$,

$$\mathcal{G}^1_{n,\ell}(i2^m+j)=\begin{cases}\mathcal{L}_i(j) & \text{if } i \text{ is even},\\ \mathcal{L}_i^{'}((j+2)\mod 2^m) & \text{if } i \text{ is odd};\end{cases}$$

$$\mathcal{G}^2_{n,\ell}(i2^m+j)=\begin{cases}\mathcal{L}_i^{'}(j) & \text{if } i \text{ is even},\\ \mathcal{L}_i((j+2)\mod 2^m) & \text{if } i \text{ is odd}.\end{cases}$$

One can see that the last item in $\mathcal{G}^1_{n,\ell}$ is $\mathcal{G}^1_{n,\ell}(2^{n-1}-1)=\mathcal{L}^{'}_{2^{\ell-2}-1}(1)$ and the last of $\mathcal{G}^2_{n,\ell}$ is $\mathcal{G}^2_{n,\ell}(2^{n-1}-1)=\mathcal{L}_{2^{\ell-2}-1}(1)$.

Now, we show that the Hamming distance between any two successive items in $\mathcal{G}^1_{n,\ell}$ or in $\mathcal{G}^2_{n,\ell}$ is ℓ.

If the two successive items are in the same list $\mathcal{L}_i$ (or $\mathcal{L}_i^{'}$), then they have Hamming distance ℓ by definition of $\mathcal{L}_i$ (or $\mathcal{L}_i^{'}$).

Otherwise, we consider two cases

- Between the item $\mathcal{G}^1_{n,\ell}(2i2^m+2^m-1)=\mathcal{L}_{2i}(2^m-1)$ and $\mathcal{G}^1_{n,\ell}((2i+1)2^m)=\mathcal{L}^{'}_{2i+1}(2)$, we have

  $$\begin{array}{lll}\mathcal{L}_{2i}(2^m-1) & =\mathcal{B}_m(2^m-1)\uplus\overline{\mathcal{B}_{\ell-1}(2i)} & =10...000\uplus\overline{\mathcal{B}_{\ell-1}(2i)},\\ \mathcal{L}^{'}_{2i+1}(2) & =I(\mathcal{B}_m(2))\uplus\mathcal{B}_{\ell-1}(2i+1) & =10...011\uplus\mathcal{B}_{\ell-1}(2i+1).\end{array}$$

  By definition of the binary reflected Gray code, we have $d_H(\mathcal{B}_m(2^m-1),I(\mathcal{B}_m(2)))=2$ and $d_H(\mathcal{B}_{\ell-1}(2i),\mathcal{B}_{\ell-1}(2i+1))=1$. Hence $d_H(\overline{\mathcal{B}_{\ell-1}(2i)},\mathcal{B}_{\ell-1}(2i+1))=\ell-2$ and we obtain

  $$d_H(\mathcal{L}_{2i}(2^m-1),\mathcal{L}^{'}_{2i+1}(2))=\ell.$$

- Between the item $\mathcal{G}^1_{n,\ell}((2i+1)2^m+2^m-1)=\mathcal{L}'_{2i+1}(1)$ and the item $\mathcal{G}^1_{n,\ell}((2i+2)2^m)=\mathcal{L}_{2i+2}(0)$. By definition we have

$$\begin{array}{lll}\mathcal{L}'_{2i+1}(1) & =I(\mathcal{B}_m(1))\uplus\overline{\mathcal{B}_{\ell-1}(2i+1)} & =10...01\uplus\overline{\mathcal{B}_{\ell-1}(2i+1)},\\ \mathcal{L}_{2i+2}(0) & =\mathcal{B}_m(0)\uplus\mathcal{B}_{\ell-1}(2i+2) & =00...00\uplus\mathcal{B}_{\ell-1}(2i+2).\end{array}$$

As for the previous case, we have $d_H(\overline{\mathcal{B}_{\ell-1}(2i+1)},\mathcal{B}_{\ell-1}(2i+2))=\ell-2$ and $d_H(I(\mathcal{B}_m(1),\mathcal{B}_m(0))=2$. Thus

$$d_H(\mathcal{L}'_{2i+1}(1),\mathcal{L}_{2i+2}(0))=\ell.$$

To prove that $\mathcal{G}^1_{n,\ell}$ is a circular code, we check that the Hamming distance between the first item $\mathcal{G}^1_{n,\ell}(0)=\mathcal{L}_0(0)$ and the last item $\mathcal{G}^1_{n,\ell}(2^{n-1}-1)=\mathcal{L}'_{2^{\ell-2}-1}(1)$ is ℓ. We have

$$\begin{array}{lll}\mathcal{L}_0(0) & =\mathcal{B}_m(0)\uplus\mathcal{B}_{\ell-1}(0) & =00...00\uplus 00...0.\\ \mathcal{L}'_{2^{\ell-2}-1}(1) & =I(\mathcal{B}_m(1))\uplus\overline{\mathcal{B}_{\ell-1}(2^{\ell-2}-1)} & =10...01\uplus 101...1.\end{array}$$

Hence, as $d_H(\mathcal{B}_m(0),I(\mathcal{B}_m(1)))=2$ and $d_H(\mathcal{B}_{\ell-1}(0),\overline{\mathcal{B}_{\ell-1}(2^{\ell-2}-1)})=\ell-2$ (since $\mathcal{B}_{\ell-1}(2^{\ell-2}-1)=010\ldots0$), we obtain $d_H(\mathcal{L}_0(0),\mathcal{L}'_{2^{\ell-2}-1}(1))=\ell$.

For the list $\mathcal{G}^2_{n,\ell}$, the same method can be used to prove that the Hamming distance is ℓ for any two successive items.

Finally, we construct the quasi (n,ℓ)-Gray code by setting:

$$\mathcal{G}_{n,\ell}(j)=\left\{\begin{array}{ll}\mathcal{G}^1_{n,\ell}(j) & \text{if } 0\le j\le 2^{n-1}-1,\\ \mathcal{G}^2_{n,\ell}((j+2)\mod 2^{n-1}) & \text{if } 2^{n-1}\le j\le 2^n-1.\end{array}\right.$$

Now, we prove that the Hamming distance between the item $\mathcal{G}_{n,\ell}(2^{n-1}-1)=\mathcal{G}^1_{n,\ell}(2^{n-1}-1)=\mathcal{L}'_{2^{\ell-2}-1}(1)$ and the item $\mathcal{G}_{n,\ell}(2^{n-1})=\mathcal{G}^2_{n,\ell}(2)=\mathcal{L}'_0(2)$ is $\ell-1$:

As we have $\mathcal{L}'_{2^{\ell-2}-1}(1)=I(\mathcal{B}_m(1))\uplus\overline{\mathcal{B}_{\ell-1}(2^{\ell-2}-1)}$ and $\mathcal{L}'_0(2)=I(\mathcal{B}_m(2))\uplus\mathcal{B}_{\ell-1}(0)$, then

$$\begin{array}{ll}d_H(\mathcal{G}_{n,\ell}(2^{n-1}-1),\mathcal{G}_{n,\ell}(2^{n-1})) & =d_H(I(\mathcal{B}_m(1)),I(\mathcal{B}_m(2)))+d_H(\overline{\mathcal{B}_{\ell-1}(2^{\ell-2}-1)},\mathcal{B}_{\ell-1}(0))\\ & =1+\ell-2=\ell-1.\end{array}$$

$\square$

Lemma 2. *For any $n\equiv 1 \mod 4$, there exists a $(n,\frac{n+1}{2})$-Gray code.*

Proof. Let $n=4q+1$ for some $q\ge 1$, $\ell=\frac{n+1}{2}=2q+1$ and let $\mathcal{G}^1_{n,\ell}$ and $\mathcal{G}^2_{n,\ell}$ as defined in the proof of Lemma 1 (for $\ell=2q+1$). In that case, we construct the $(4q+1,2q+1)$-Gray code by setting:

$$\mathcal{G}_{n,\ell}(j)=\left\{\begin{array}{ll}\mathcal{G}^1_{n,\ell}(j) & \text{if } 0\le j\le 2^{4q}-1,\\ \mathcal{G}^2_{n,\ell}((j+r)\mod 2^{4q}) & \text{if } 2^{4q}\le j\le 2^{4q+1}-1,\end{array}\right.$$

with $r=\frac{1}{3}(2^{2q+1}+1)$.

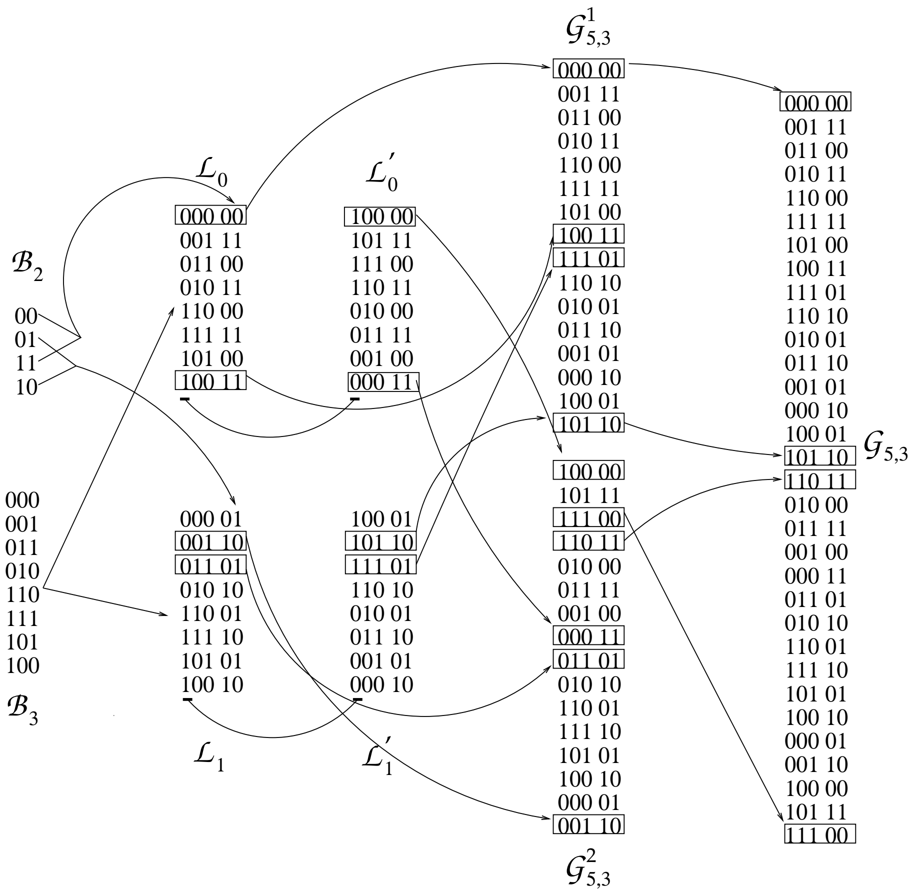

Figure 2: The $(5,3)$-Gray code $\mathcal{G}_{5,3}$.

Now, we prove that the Hamming distance between the item $\mathcal{G}_{n,\ell}(2^{4q}-1) = \mathcal{G}^1_{n,\ell}(2^{4q}-1) = \mathcal{L}'_{2^{2q-1}-1}(2^{2q+1}-1)$ and the item $\mathcal{G}_{n,\ell}(2^{4q}) = \mathcal{G}^2_{n,\ell}(r) = \mathcal{L}'_0(r)$ is $2q+1$: One can see that

$$\mathcal{L}'_{2^{2q-1}-1}(2^{2q+1}-1) \quad = I(\mathcal{B}_{2q+1}(2^{2q+1}-1)) \uplus \overline{\mathcal{B}_{2q}(2^{2q-1}-1)} \quad = \underbrace{10\ldots01}_{2q+1}\underbrace{101\ldots1}_{2q},$$

and

$$\mathcal{L}'_0(r) \quad = I(\mathcal{B}_{2q+1}(r)) \uplus \overline{\mathcal{B}_{2q}(0)} \quad = \underbrace{1\ldots10}_{2q+1}\underbrace{1\ldots1}_{2q}.$$

The fact that $I(\mathcal{B}_{2q+1}(r)) = \underbrace{1\ldots10}_{2q+1}$, or equivalently, that $\mathcal{B}_{2q}(r) = \underbrace{1\ldots10}_{2q}$ can be proved by induction on q: For $q=1$, we have $\mathcal{B}_2(3) = 10$. Suppose that $\mathcal{B}_{2q}(\frac{1}{3}(2^{2q+1}+1)) = 1\ldots10$ for some $q>1$. By construction of the binary reflected Gray code, we have $\mathcal{B}_{n+1}(2^{n+1}-j) = 1 \uplus \mathcal{B}_n(j)$ for all $j, 0 \le j \le 2^n-1$. Then,

$$\mathcal{B}_{2q+1}(2^{2q+1} - \frac{1}{3}(2^{2q+1}+1)) = 1 \uplus \mathcal{B}_{2q}(\frac{1}{3}(2^{2q+1}+1)) = 1\ldots10.$$

Hence,

$$\mathcal{B}_{2q+2}(2^{2q+2} - 2^{2q+1} + \frac{1}{3}(2^{2q+1}+1)) = 1 \uplus \mathcal{B}_{2q+1}(2^{2q+1} - \frac{1}{3}(2^{2q+1}+1)) = 1\ldots10.$$

Since $2^{2q+2} - 2^{2q+1} + \frac{1}{3}(2^{2q+1}+1) = 2^{2q+1} + \frac{1}{3}(2^{2q+1}+1) = \frac{1}{3}(2^{2(q+1)+1}+1)$, we have shown that $\mathcal{B}_{2q+2}(\frac{1}{3}(2^{2(q+1)+1}+1)) = 1\ldots10$.

Consequently,

$$d_H(\mathcal{L}'_{2^{2q-1}-1}(2^{2q+1}-1), \mathcal{L}'_0(r)) = 2q+1.$$

□

The construction of a $(5,3)$-Gray code is illustrated in Figure 2.

To end this section, we want to mention that it seems possible to construct a (n,ℓ)-Gray code for any $n > 1$ and any odd $\ell < n$, but we did not try to go further in this direction since quasi (n,ℓ)-Gray codes are sufficient for us when $n \not\equiv 1 \bmod 4$.

3 The radio antipodal and radio numbers of the hypercube

We begin by showing a lower bound for the radio k-chromatic number of the hypercube. The argument is similar with the one used by Liu and Justie [12] for obtaining a lower bound on the radio antipodal number of the cycle.

Lemma 3. *For any three vertices x, y and z of the hypercube Q_n,*

$$d(x,y) + d(y,z) + d(x,z) \leq 2n.$$

Moreover, if $y = \overline{x}$ then

$$d(x,y) + d(y,z) + d(x,z) = 2n.$$

Proof. The first inequality easily follows from the fact that for a given index $i, 0 \leq i \leq n-1$, any three binary n-bit strings have at most 2 bits in common on that index i.

If $y = \overline{x}$, then one can see that $d(x,y) = n$ and $d(y,z) + d(x,z) = n$. □

Proposition 1. *For any positive integer n,*

$$rc_k(Q_n) \geq (2^{n-1}-1)\lceil \frac{3k-2n+3}{2} \rceil + k - n + 1.$$

Proof. Let f be a radio k-labeling of Q_n and $x_0, x_1, \ldots, x_{2^n-1}$ be an ordering of the vertices such that $f(x_{i+1}) \geq f(x_i)$, $0 \leq i \leq 2^n-2$.

By definition, we have

$$\begin{array}{ll} f(x_{i+2}) - f(x_{i+1}) & \geq k+1-d(x_{i+2},x_{i+1}), \\ f(x_{i+1}) - f(x_i) & \geq k+1-d(x_{i+1},x_i), \\ f(x_{i+2}) - f(x_i) & \geq k+1-d(x_{i+2},x_i). \end{array} \quad (1)$$

Summing up these three inequalities and using Lemma 3, we obtain :

$$2f(x_{i+2})-2f(x_i) \geq 3k+3-d(x_{i+2},x_{i+1})-d(x_{i+1},x_i)-d(x_{i+2},x_i) \geq 3k+3-2n,$$

and thus

$$f(x_{i+2}) - f(x_i) \geq \lceil \frac{3k+3-2n}{2} \rceil. \tag{2}$$

By applying for all values of i, $0 \leq i \leq 2^n - 3$, we obtain

$$\begin{aligned} \sum_{i=0}^{2^n-3} (f(x_{i+2}) - f(x_i)) &\geq \sum_{i=0}^{2^n-3} (\lceil \tfrac{3k+3-2n}{2} \rceil) \quad \Leftrightarrow \\ f(x_{2^n-1}) + f(x_{2^n-2}) - f(x_1) - f(x_0) &\geq \sum_{i=0}^{2^n-3} \lceil \tfrac{3k+3-2n}{2} \rceil. \end{aligned}$$

Thus, assuming $f(x_0) = 0$ and since $f(x_{2^n-2}) \leq f(x_{2^n-1})-k-1+d(x_{2^n-2},x_{2^n-1})$ and $f(x_1) \geq f(x_0) + k + 1 - d(x_1, x_0)$, we get

$$2f(x_{2^n-1}) \geq (2^n - 2)\lceil \frac{3k+3-2n}{2} \rceil + 2k + 2 - d(x_1, x_0) - d(x_{2^n-2}, x_{2^n-1}).$$

Consequently, as $d(x_1, x_0) \leq n$ and $d(x_{2^n-2}, x_{2^n-1}) \leq n$, we obtain $2f(x_{2^n-1}) \geq (2^n - 2)\lceil \frac{3k+3-2n}{2} \rceil + 2k - 2n + 2$.

Therefore,

$$\mathrm{rc}_k(Q_n) \geq (2^{n-1} - 1)\lceil \frac{3k-2n+3}{2} \rceil + k - n + 1.$$

□

3.1 The radio antipodal number

In this section, we prove Theorem 2 by showing a lower bound and a upper bound that are equal.

Theorem 4. *For any positive integer n,*

$$\mathrm{an}(Q_n) \geq (2^{n-1} - 1)\lceil \frac{n}{2} \rceil + \varepsilon(n),$$

with $\varepsilon(n) = 1$ if $n \equiv 0 \bmod 4$, $\varepsilon(n) = 0$ otherwise.

Proof. Using Proposition 1 with $k = \mathrm{diam}(Q_n) - 1 = n - 1$, we obtain

$$\mathrm{an}(Q_n) \geq (2^{n-1} - 1)\lceil \frac{n}{2} \rceil.$$

Now, for $n \equiv 0 \bmod 4$, we show that $\mathrm{an}(Q_n) \geq (2^{n-1} - 1)\lceil \frac{n}{2} \rceil + 1$. Let f be a radio antipodal labeling of Q_n and $x_0, x_1, \ldots, x_{2^n-1}$ be an ordering of the vertices such that $f(x_{i+1}) \geq f(x_i)$, $0 \leq i \leq 2^n - 2$. By contradiction, assume that $\mathrm{an}(Q_n) = (2^{n-1} - 1)\lceil \frac{n}{2} \rceil$. Then the inequalities (1) and (2) in proof of Proposition 1 must become equalities and we must have $d(x_1, x_0) = d(x_{2^n-2}, x_{2^n-1}) = n$.

By Lemma 3, as $x_1 = \overline{x_0}$, then $d(x_2, x_1) = d(x_2, x_0) = \frac{n}{2}$ since $f(x_2) - f(x_0) = \frac{n}{2} = n - d(x_2, x_0)$. By combining the equalities (1) for $i = 1$, we obtain

$d(x_3, x_2) = n$ and thus, by Lemma 3, $d(x_4, x_3) = d(x_4, x_2) = \frac{n}{2}$. Applying this for all $i, 2 \leq i \leq 2^n - 3$, we can see that $d(x_{2i}, x_{2i-1}) = \frac{n}{2}$ and $d(x_{2i+1}, x_{2i}) = \frac{n}{2}$, $0 \leq i \leq 2^{n-1} - 1$. Such an ordering of the vertices of Q_n is equivalent to a $(n-1, \frac{n}{2})$-Gray code, which does not exist since $\frac{n}{2}$ is even (as we have seen in Remark 1), thus a contradiction. $\square$

Theorem 5. *For any positive integer $n \geq 1$,*

$$\mathrm{an}(Q_n) \leq (2^{n-1} - 1)\lceil \frac{n}{2} \rceil + \varepsilon(n),$$

with $\varepsilon(n) = 1$ if $n \equiv 0 \bmod 4$, $\varepsilon(n) = 0$ otherwise.

Proof. Let Q^1_{n-1} and Q^2_{n-1} be respectively the two sub-hypercubes of Q_n induced by the strings with a 0 on the left (1 on the left, respectively).

Let $x_0, x_1, x_2, \ldots, x_{2^{n-1}-1}$ be the ordering of the vertices of Q^1_{n-1} induced by a (quasi if $n \not\equiv 2 \bmod 4$) $(n-1, \lfloor \frac{n}{2} \rfloor)$-Gray code $\mathcal{G}_{n-1, \lfloor \frac{n}{2} \rfloor}$ (such a code exists by Lemma 1 and Lemma 2): $x_i = 0 \uplus \mathcal{G}_{n-1, \lfloor \frac{n}{2} \rfloor}(i), 0 \leq i \leq 2^n - 1$.

Let $y_i = \overline{x_i} = 1 \uplus \overline{\mathcal{G}_{n-1, \lfloor \frac{n}{2} \rfloor}(i)}, 0 \leq i \leq 2^n - 1$. Hence $y_0, y_1, y_2, \ldots, y_{2^{n-1}-1}$ is an ordering of the vertices Q^2_{n-1} induced by a $(n-1, \lfloor \frac{n}{2} \rfloor)$-Gray code.

We consider two cases depending on whether n is even or odd.

Case 1. $n = 2p + 1$ is odd, for some $p \geq 1$.

We define a labeling function f on Q_{2p+1} as follows

$$\begin{cases} f(x_0) = 0, & \\ f(x_{i+1}) = f(x_i) + p + 1 & 0 \leq i \leq 2^{2p} - 2 \\ f(y_i) = f(x_i), & 0 \leq i \leq 2^{2p} - 1. \end{cases}$$

Then the two vertices $x_{2^{2p}-1}$ and $y_{2^{2p}-1}$ have the maximum label :

$$f(x_{2^{2p}-1}) = f(y_{2^{2p}-1}) = \sum_{i=1}^{2^{2p}-1} (p+1) = (2^{2p} - 1)(p + 1).$$

Now, we show that the distance condition is verified for any two vertices a and b of Q_{2p+1}. We distinguish two cases:

Subcase 1.1 The two vertices are both in Q^1_{2p} (or in Q^2_{2p}, because we have $d(x_i, x_j) = d(y_i, y_j)$, $f(x_i) = f(y_i)$ and $f(x_j) = f(y_j)$): assume $a = x_i$ and $b = x_j$, with $i > j$. Then

$$\begin{array}{rll} |f(x_i) - f(x_j)| & \geq \mathrm{diam}(Q_{2p+1}) - d(x_i, x_j) & \Leftrightarrow \\ |f(x_j) + (i-j)(p+1) - f(x_j)| & \geq 2p + 1 - d(x_i, x_j) & \Leftrightarrow \\ (i-j)(p+1) & \geq 2p + 1 - d(x_i, x_j). & \end{array}$$

If $i > j + 1$, then the inequality is clearly satisfied and if $i = j + 1$, it is also satisfied since $d(x_j, x_{j+1}) \geq p$.

Subcase 1.2 The vertex $a = x_i$ is in Q^1_{2p} and the vertex $b = y_j$ is in Q^2_{2p}. As $y_j = \overline{x_j}$, then we have $d(x_i, y_j) = 2p + 1 - d(x_i, x_j)$. Thus,

$$\begin{array}{rll} |f(x_i) - f(y_j)| & \geq \mathrm{diam}(Q_{2p+1}) - d(x_i, y_j) & \Leftrightarrow \\ |f(x_i) - f(x_j)| & \geq 2p + 1 - (2p + 1 - d(x_i, x_j)) & \Leftrightarrow \\ |f(x_i) - f(x_j)| & \geq d(x_i, x_j) & \Leftrightarrow \\ (i - j)(p + 1) & \geq d(x_i, x_j). & \end{array}$$

If $i = j + 1$, then $d(x_i, x_j) \leq p + 1$. Hence the inequality is satisfied. If $i > j + 1$, then $d(x_i, x_j) \leq 2p$. Thus, we obtain $(i - j)(p + 1) \geq 2p$ which is true for any $p \geq 1$ and any $i > j + 1$.

Consequently, f is an antipodal labeling of Q_{2p+1} and

$$\mathrm{an}(Q_{2p+1}) \leq (2^{2p} - 1)(p + 1).$$

Case 2. $n = 2p$ is even, for some $p \geq 1$.

We define a labeling function f on Q_{2p} as follows

$$\begin{cases} f(x_0) = 0, & \\ f(x_{i+1}) = f(x_i) + p, & \text{with } 0 \leq i \leq 2^{2p-1} - 2, i \neq 2^{2p-2} - 1 \text{ if } p \equiv 0 \bmod 2, \\ f(x_{2^{2p-2}}) = f(x_{2^{2p-2}-1}) + p + 1, & \text{if } p \equiv 0 \bmod 2, \\ f(y_i) = f(x_i), & 0 \leq i \leq 2^{2p-1} - 1. \end{cases}$$

Then the two vertices $x_{2^{2p-1}-1}$ and $y_{2^{2p-1}-1}$ have the maximum label : If $p \equiv 0 \bmod 2$, then $f(x_{2^{2p-1}-1}) = f(y_{2^{2p-1}-1}) = 1 + \sum_{i=1}^{2^{2p}-1} (p) = (2^{2p} - 1)p + 1$ and if $p \equiv 1 \bmod 2$, then $f(x_{2^{2p-1}-1}) = f(y_{2^{2p-1}-1}) = \sum_{i=1}^{2^{2p}-1} (p) = (2^{2p} - 1)p$.

The fact that f is an antipodal labeling can be shown as for the previous case. Consequently,

$$\mathrm{an}(Q_{2p}) \leq (2^{2p-1} - 1)p + \varepsilon(2p),$$

with $\varepsilon(n) = 1$ if $n \equiv 0 \bmod 4$, $\varepsilon(n) = 0$ otherwise. □

3.2 The radio number

In this section, we prove Theorem 3 presented in the introduction, which we recall below:

Theorem 6. *For any positive integer $n \geq 1$,*

$$\mathrm{rn}(Q_n) = (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1.$$

Proof. The fact that $\mathrm{rn}(Q_n) \geq (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1$ is a direct consequence of Proposition 1 for $k = \mathrm{diam}(Q_n) = n$.

Now, to show that $\mathrm{rn}(Q_n) \leq (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1$, we use the same ordering $x_0, x_1, x_2, \ldots, x_{2^{n-1}-1}$ and $y_0, y_1, y_2, \ldots, y_{2^{n-1}-1}$ of Q_n used in the proof of Theorem 5.

We define a labeling function f on Q_n as follows :

$$\begin{cases} f(x_0) = 0, & \\ f(x_{i+1}) = f(x_i) + \lceil \frac{n+3}{2} \rceil, & 0 \leq i \leq 2^{n-1} - 2, \\ f(y_i) = f(x_i) + 1, & 0 \leq i \leq 2^{n-1} - 1. \end{cases}$$

Then the vertex $y_{2^{n-1}-1}$ has the maximum label :

$$f(y_{2^{n-1}-1}) = f(x_{2^{n-1}-1}) + 1 = (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1.$$

As for the radio antipodal number, to show that the distance condition is verified for any two vertices a and b of Q_n, we distinguish two cases:

Case 1. The two vertices are both in Q^1_{n-1} (or in Q^2_{n-1}): assume $a = x_i$ and $b = x_j$, with $i > j$. Then

$$\begin{array}{rll} |f(x_i) - f(x_j)| & \geq \operatorname{diam}(Q_n) + 1 - d(x_i, x_j) & \Leftrightarrow \\ |f(x_j) + (i-j)\lceil \frac{n+3}{2} \rceil - f(x_j)| & \geq n + 1 - d(x_i, x_j) & \Leftrightarrow \\ (i-j)\lceil \frac{n+3}{2} \rceil & \geq n + 1 - d(x_i, x_j). & \end{array}$$

If $i > j + 1$, then the inequality is clearly satisfied and if $i = j + 1$, it is also satisfied since $d(x_j, x_{j+1}) \geq \lfloor \frac{n}{2} \rfloor$.

Case 2. The vertex $a = x_i$ is in Q^1_n and the vertex $b = y_j$ is in Q^2_n. As, $y_j = \overline{x_j}$ then we have $d(x_i, y_j) = n - d(x_i, x_j)$ Thus,

$$\begin{array}{rll} |f(x_i) - f(y_j)| & \geq \operatorname{diam}(Q_n) + 1 - d(x_i, y_j) & \Leftrightarrow \\ |f(x_i) - f(x_j)| & \geq n + 1 - (n - d(x_i, x_j)) & \Leftrightarrow \\ |f(x_i) - f(x_j)| & \geq 1 + d(x_i, x_j) & \Leftrightarrow \\ (i-j)\lceil \frac{n+3}{2} \rceil & \geq 1 + d(x_i, x_j). & \end{array}$$

If $i = j + 1$, then $d(x_i, x_j) \leq \lfloor \frac{n}{2} \rfloor + 1$. Hence the inequality is satisfied. If $i > j + 1$, then $d(x_i, x_j) \leq n - 1$. Thus, we obtain $(i - j)\lceil \frac{n+3}{2} \rceil \geq n$ which is true for any $i > j + 1$.

Consequently, f is a radio labeling of Q_n and $\operatorname{rn}(Q_n) \leq (2^{n-1} - 1)\lceil \frac{n+3}{2} \rceil + 1$. □

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