Total and Fractional Total Colourings of Circulant Graphs (preliminary version)

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Abstract

In this paper, the total chromatic number and fractional total chromatic number of circulant graphs are studied. For cubic circulant graphs we give upper bounds on the fractional total chromatic number and for 4-regular circulant graphs we find the total chromatic number for some cases and we give the exact value of the fractional total chromatic number in most cases.

Keywords: Graph colouring; Total colouring; Fractional total colouring ; Circulant graph.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $\Delta(G)$ the maximum degree of the graph G.

The notion of total colouring is a mixing of the vertex colouring and edge colouring; it was introduced and studied by Behzad [1] and Vizing [14]. We call a vertex or an edge an *element* of the graph and two elements are *neighbors* if they are either adjacent or incident.

Definition 1 *A* proper total k-colouring *of a graph G is a mapping f from the element set $V(G) \cup E(G)$ to the colour set $C = \{0, 1, 2, ..., k-1\}$, such that any two neighboring elements receive distinct colours. The* total chromatic number $\chi^{''}(G)$ *of G is the smallest positive integer k for which there exists a proper total k-colouring.*

Clearly, for any graph G, $\chi^{''}(G) \geq \Delta(G) + 1$, since a vertex of maximum degree needs a different colour from those $\Delta(G)$ assigned to its incident edges. The problem of computing the total chromatic number of a given graph is NP-complete even where restricted to regular bipartite graphs [11, 12].

Behzad [1] and Vizing [14] independently made the following conjecture for the total colouring of a graph.

Total Colouring Conjecture (TCC): For any graph G,

$$\chi^{''}(G) \leq \Delta(G) + 2.$$

In general the total colouring conjecture implies that for every graph G, $\chi^{''}(G)$ attains one of the two values $\Delta(G) + 1$ or $\Delta(G) + 2$. A graph G is *type 1* if $\chi^{''}(G) = \Delta(G) + 1$ and it is *type 2* if $\chi^{''}(G) = \Delta(G) + 2$.

The TCC is known to hold for some classes of graphs: bipartite graphs, some planar graphs [3], graphs of maximum degree at least 5 [9]. Moreover, the exact value of the total chromatic number is known for some simple classes of graphs: the path P_n, $n \geq 3$ is type 1; the cycle C_n is type 1 if $n \equiv 0 \bmod 3$ and is type 2 otherwise [15]; the complete bipartite graph $K_{n,m}$ is type 1 if $n = m$ and is type 2 if $n \neq m$ [2]; the complete graph K_n is type 1 if n is odd and is type 2 if n is even [2].

For a sequence of positive integers $1 \leq d_1 < d_2 < \ldots < d_\ell \leq \lfloor \frac{n}{2} \rfloor$ the circulant graph $G = C_n(d_1, d_2, \ldots, d_\ell)$ has vertex set $V = Z_n = \{0, 1, 2, \ldots, n-1\}$, two vertices x, y being adjacent iff $x \equiv (y \pm d_i) \bmod n$ for some i, $1 \leq i \leq \ell$.

We will use the following notation for the circulant graph $G = C_n(d_1, d_2, \ldots, d_\ell)$: $V(G) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and $E(G) = \bigcup_{i=1}^{\ell} E_i(G)$ where $E_i(G) = \{e_0^i, e_1^i, \ldots, e_{n-1}^i\}$ and $e_j^i = (v_j, v_{(j+d_i) \bmod n})$ for $1 \leq i \leq \ell$ and $0 \leq j \leq n-1$ (if n is even and $d_\ell = \frac{n}{2}$, then $E_\ell(G) = \{e_0^\ell, e_1^\ell, \ldots, e_{\frac{n-1}{2}}^\ell\}$). An edge of $E_i(G)$ will be called an edge of *length* d_i.

Concerning total colouring of circulant graphs, little is known : cubic circulant graphs were studied by Hackmann and Kemnitz [6] who determine their type and a upper bound on the circular chromatic number of cubic circulant graphs of type 2. Campos and Mello [4, 5] studied the power of cycle C_n^p. For $p = 2$ they showed that if $n = 7$ then C_n^2 is type 2, and type 1 otherwise. In general they showed that $\chi''(C_n^p) \leq \Delta(C_n^p) + 2$ and conjectured that C_n^p is type 2 if $p > \frac{n}{3} - 1$ and n is odd; and type 1 otherwise.

An independent total set in a graph is a subset of mutually non neighboring elements of G. The *total independence number* of G denoted by $\alpha''(G)$ is the size of a largest total independent set in G.

Definition 2 *A* b-fold total colouring *of a graph* G *is an assignment of sets of size* b *to the elements of a graph such that neighboring elements receive disjoint sets. An* a/b*-total colouring is a* b*-fold total colouring out of* a *available colours.*

The fractional total chromatic number $\chi_f''(G)$ *is given by*

$$\chi_f''(G) = min\{\frac{a}{b},\ G \textit{ has an } a/b\textit{-total colouring}\}.$$

The fractional chromatic number and the fractional chromatic index of G are denoted by $\chi_f(G)$ and $\chi_f'(G)$, respectively.

Property 1 *For every graph* G *of order* n*,*

$$\chi_f''(G) \geq \frac{|V(G)| + |E(G)|}{\alpha''(G)}.$$

The fractional version of the TCC was proved by Kilakos and Reed [8]: for any graph G, $\chi_f''(G) \leq \Delta(G) + 2$. However, the complexity of finding the fractional total chromatic number it still unknown, and there is not a lot of research concerning fractional total colourings.

The *circular total chromatic number*, denoted by $\chi_c''(G)$, is the circular extension of total colouring which was considered in [7, 6].

Property 2 *For every graph* G *of order* n*,*

$$\Delta(G) + 1 \leq \chi_f''(G) \leq \chi_c''(G) \leq \chi''(G).$$

In this paper, we first study in Section 2 the fractional total chromatic number for circulant graph $C_n(d_1, d_2, \ldots, d_\ell)$ by introducing the notion of fractional stable in circulant graphs, and we give a general upper bound on the fractional total chromatic number. In Section 3, we investigate and determine the fractional total chromatic number of cubic circulant graphs. Finally in Section 4, we give the exact value of the total chromatic number for some 4-regular circulant graphs and we determine the fractional total chromatic number of 4-regular circulant graphs in most cases.

2 Fractional stables in circulant graphs

In this section we define the notion of *fractional stable* of a circulant graph and we give a general upper bound on the fractional total chromatic number of circulant graphs.

In the rest of the paper, by a *stable* of a graph G we mean an independent total set (a subset of independent elements of $V(G) \cup E(G)$).

Definition 3 *For a stable S of $G_n(d_1, d_2, \ldots, d_\ell)$, we denote by $V(S)$ the vertices of S and by $E_i(S)$ the edges of length d_i of S. Let also $n(S) = |V(S)|$ and $m_i(S) = |E_i(S)|$ for $1 \leq i \leq \ell$. Such a stable S will also be called a $(n(S), m_1(S), m_2(S), \ldots, m_\ell(S))$-stable.*

Definition 4 *A (multi) set $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ of k stables of $C_n(d_1, d_2, \ldots, d_\ell)$ form a fractional $(\eta, \mu_1, \mu_2, \ldots, \mu_\ell)$-stable $F_{\mathcal{S}}$ with*

$$\begin{cases} \eta = \sum\limits_{i=1}^{k} \frac{n(S_i)}{k}, \\ \mu_j = \sum\limits_{i=1}^{k} \frac{m_j(S_i)}{k}, \quad 1 \leq j \leq \ell. \end{cases}$$

Definition 5 *A $(\eta, \mu_1, \mu_2, \ldots, \mu_\ell)$-stable (fractional or not) in $C_n(d_1, d_2, \ldots, d_\ell)$ is* complete *if and only if $\eta + 2\sum_{i=1}^{\ell} \mu_i = n$.*

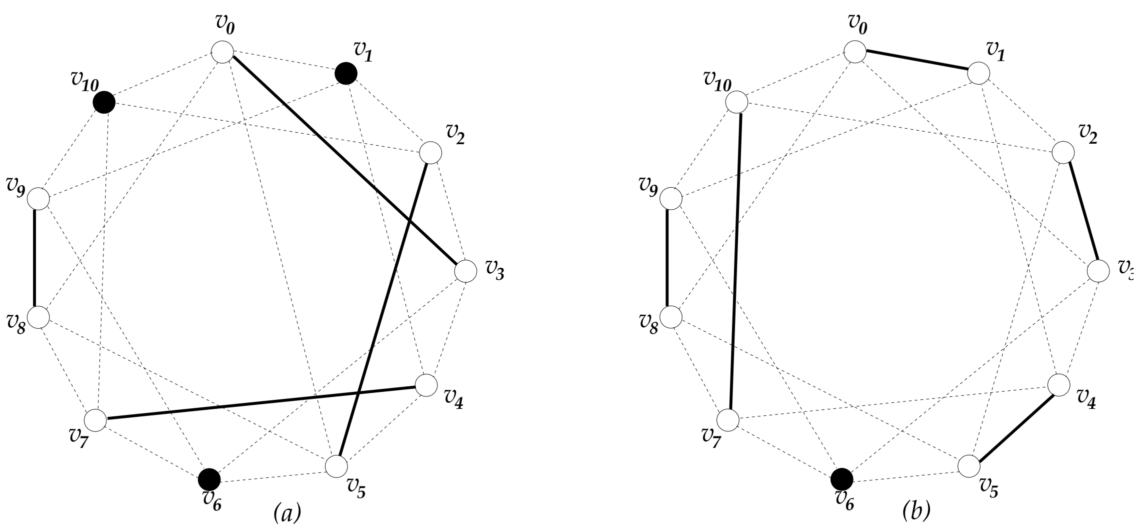

Figure 1: A $(3,1,3)$-stable *(a)* and a $(1,4,1)$-stable *(b)* in $C_{11}(1,3)$.

Definition 6 *A $(\eta, \mu_1, \mu_2, \ldots, \mu_\ell)$-stable (fractional or not) in $C_n(d_1, d_2, \ldots, d_\ell)$ is* balanced *if*

$$\begin{cases} \eta = \mu_i & 1 \leq i \leq \ell & \text{if } d_\ell \leq \lfloor \frac{n-1}{2} \rfloor, \\ \eta = \mu_i = 2\mu_\ell & 1 \leq i \leq \ell - 1 & \text{if } d_\ell = \frac{n}{2}. \end{cases}$$

Figure 1 shows an example of a $(3,1,3)$-stable (part *(a)*) and a $(1,4,1)$-stable (part *(b)*) in $C_{11}(1,3)$, whose union forms a fractional $(2, \frac{5}{2}, 2)$-stable (i.e. complete but not balanced). On the other hand we can form a balanced and complete fractional $(\frac{11}{5}, \frac{11}{5}, \frac{11}{5})$-stable from 3 copies of the stable depicted on part *(b)* and 2 copies of the stable depicted on part *(a)*.

Note that :

$$\chi_f(C_n) = \chi_f'(C_n) = \begin{cases} 2 & \text{if } n = 2p, \\ \frac{2p+1}{p} & \text{if } n = 2p+1. \end{cases}$$

Lemma 1 *Let $G = C_n(d_1, d_2, \ldots, d_\ell)$ be a circulant graph and let $F_{\mathcal{S}}$ be a fractional $(\eta, \mu_1, \mu_2, \ldots, \mu_\ell)$-stable. Then the fractional total chromatic number of G satisfies:*

- *If $d_\ell \leq \lfloor \frac{n-1}{2} \rfloor$ and $\eta \geq \max_{i=1}^{\ell}(\mu_i)$, then*

$$\chi_f''(G) \leq \frac{n}{\eta} + \sum_{i=1}^{\ell} (\frac{\eta - \mu_i}{\eta}) \chi_f'(C_{p_i}), \text{ where } p_i = \frac{n}{\gcd(n, d_i)}.$$

- *If $d_\ell = \frac{n}{2}$ and $\eta \geq \max\{\mu_1, \mu_2, \ldots, \mu_{\ell-1}, 2\mu_\ell\}$, then*

$$\chi_f''(G) \leq \frac{n}{\eta} + \sum_{i=1}^{\ell-1} (\frac{\eta - \mu_i}{\eta}) \chi_f'(C_{p_i}) - \frac{2\mu_\ell}{\eta} + 1, \text{ where } p_i = \frac{n}{\gcd(n, d_i)}.$$

Proof : Assume that $G = C_n(d_1, d_2, \ldots, d_\ell)$ admits a fractional $(\eta, \mu_1, \mu_2, \ldots, \mu_\ell)$-stable $F_{\mathcal{S}}$ obtained from k stables $S_1^0, S_2^0, \ldots, S_k^0$, with $|S_i^0| > 0$, $\eta = \sum_{i=1}^{k} \frac{n(S_i^0)}{k}$ and $\mu_j = \sum_{i=1}^{k} \frac{m_j(S_i^0)}{k}$ for $1 \leq j \leq \ell$.

Let S_i^θ be the stable obtained from S_i^0 by a 'rotation' of $\frac{2\pi\theta}{n}$ i.e. if $v_r \in V(S_i^0)$ then $v_{r+\theta} \in V(S_i^\theta)$ and if $e_r^q \in E(S_i^0)$ the $e_{r+\theta}^q \in E(S_i^\theta)$ with $0 \leq \theta \leq n-1$, $1 \leq q \leq \ell$, $0 \leq i \leq k$ and addition taken modulo n.

We first prove the first assertion. One can see that each vertex v_r is exactly in $s = \sum_{i=1}^{k} n(S_i^0) = k\eta$ stables and each edge e_r^j of length d_j is exactly in $a_j = \sum_{i=1}^{k} m_j(S_i^0) = k\mu_j$ stables with $1 \leq j \leq \ell$ and $0 \leq r \leq n-1$. For instance if $\{v_{t_1}, v_{t_2}, \ldots, v_{t_{n(S_i^0)}}\} \subseteq S_i^0$ then $v_r \in V(S_i^\theta)$ for $\theta = r - t_1, r - t_2, \ldots, r - t_{n(S_i^0)}$ (modulo n) and the same goes for the edges of each length.

Therefore, assigning a colour c_i^j to all elements of each stable S_i^j result in a multi-colouring of G with kn colours such that each vertex has s colours and each edge of length d_i has a_i colours.

Now, since the edges of length d_i of G induce $\gcd(n, d_i)$ cycles of length $\frac{n}{\gcd(n,d_i)}$, and assuming $\chi_f'(C_{\frac{n}{\gcd(n,d_i)}}) = \frac{v}{w}$ for some integers v and w, there exists a $w(s - a_i)$-fold colouring of the edges of length d_i with $v(s - a_i)$ colours and by the above, there also exists a multi-colouring of G with knw colours such that each vertex gets sw colours and each edge of length d_i gets $a_i w$ colours. Combining these two colourings, we obtain a sw-fold total colouring of G with $knw + \sum_{i=1}^{k} v(s - a_i)$ colours.

Hence, as we have $s = k\eta$ and $a_j = \sum_{i=1}^{k} m_j(S_i) = k\mu_j$ for $1 \leq j \leq \ell$, we obtain a fractional colouring with $\frac{knw}{sw} + \frac{\sum_{i=1}^{k} v(s-a_i)}{sw} = \frac{kn}{k\eta} + \frac{\sum_{i=1}^{k}(k\eta - k\mu_i)v}{k\eta w} = \frac{n}{\eta} + \sum_{i=1}^{\ell} (\frac{\eta - \mu_i}{\eta}) \chi_f'(C_{\frac{n}{\gcd(n,d_i)}})$ colours.

To prove the second assertion of the lemma we proceed in a similar way except that we have in this case $a_\ell = 2\sum_{i=1}^{k} m_\ell(S_i^0)$. As all the edges of length $d_\ell = \frac{n}{2}$ form an independent set (thus can be coloured with the same $(s - a_i)$ colours), we obtain in this case

$$\chi_f''(G) \leq \frac{n}{\eta} + \sum_{i=1}^{\ell-1} (\frac{\eta - \mu_i}{\eta}) \chi_f'(C_{\frac{n}{\gcd(n,d_i)}}) - \frac{2\mu_\ell}{\eta} + 1. \square$$

$\square$

If we have $\mu_j = \eta$ for all j, $1 \leq j \leq \ell$, then we obtain the following corollary:

Corollary 1 *Let p be a positive rational number. If $G = C_n(d_1, d_2, \ldots, d_\ell)$ possesses a fractional $(p, p, p, \ldots, p)$-stable with $d_\ell \leq \lfloor \frac{n-1}{2} \rfloor$, then the fractional total chromatic number of G satisfies:*

$$\chi_f''(G) \leq \frac{n}{p}.$$

Lemma 2 *For the circulant graph $G = C_n(d_1, d_2, \ldots, d_\ell)$, the following assertions are equivalent:*

i) $\chi_f''(G) = \Delta(G) + 1$,

ii) G possesses a complete balanced fractional stable.

Proof : To prove this lemma it is easier to use the original definition of the fractional total colouring rather than the (equivalent) definition given in the introduction: a fractional total colouring of a graph G in $\Delta(G)+1$ colours is a function f from $V(G)\cup E(G)$ to $[0,1]$ such that $\sum_{S\in\mathcal{I}} f(S)=\Delta(G)+1$ and $\forall x\in V(G)\cup E(G)$, $\sum_{S\in\mathcal{I},x\in S} f(S)\geq 1$, where $\mathcal{I}$ is the set of all total stables of G.

Suppose that $\mathcal{I}^+=\{I_1,I_2,\ldots,I_t\}$ is the subset of $\mathcal{I}$ such that we have $f(I)=0$ for all $I\notin\mathcal{I}\setminus\mathcal{I}^+$. By allowing multiple copies of a stable S in $\mathcal{I}^+$, we can assume that $f(I_i)=\frac{\Delta(G)+1}{t}$ and thus we see that each element is at least in α stables, with $\alpha\geq\lceil\frac{t}{\Delta(G)+1}\rceil$.

This implies that $\sum_{i=1}^{t} n(I_i)\geq n\alpha\geq n\lceil\frac{t}{\Delta(G)+1}\rceil$ and $\forall j$, $\sum_{i=1}^{t} m_j(I_i)\geq n\lceil\frac{t}{\Delta(G)+1}\rceil$.

On the other part we have by definition $\forall i$, $n(I_i)+2\sum_{j=1}^{\ell} m_j(I_i)\leq n$, thus

$$\sum_{i=1}^{t}(n(I_i)+2\sum_{j=1}^{\ell} m_j(I_i))\leq tn.$$

Together we obtain $(\Delta(G)+1)n\lceil\frac{t}{\Delta(G)+1}\rceil\leq\sum_{i=1}^{t}(n(I_i)+2\sum_{j=1}^{\ell} m_j(I_i))\leq tn$, and

$$\sum_{i=1}^{t}(n(I_i)+2\sum_{j=1}^{\ell} m_j(I_i)) \quad = \sum_{i=1}^{t} n(I_i)+2\sum_{i=1}^{t}\sum_{j=1}^{\ell} m_j(I_i).$$

Therefore, we must have $n(I_i)+2\sum_{j=1}^{\ell} m_j(I_i)=n$ for all i (each stable I_i is complete) and $\sum_{i=1}^{t} n(I_i)=\sum_{i=1}^{t} m_j(I_i)$ (the fractional stable formed by the set of stables $\mathcal{I}^+$ is balanced).

To show that the converse is true, assume that G possesses a complete balanced fractional stable $F_{\mathcal{S}}$. Then $F_{\mathcal{S}}$ is a fractional $(p,p,\ldots,p)$-stable with $p=\frac{n}{2\ell+1}$ and Corollary 1 gives that $\chi_f''(G)=2\ell+1=\Delta(G)+1$. □ □

3 Cubic circulant graphs

In this section we study the fractional chromatic number of cubic circulant graphs.

The only cubic circulant graphs are of the form $G=C_{2p}(d,p)$. Hackmann and Kemnitz [6] showed the following results:

Lemma 3 ([6]) *If ℓ is the greatest common divisor of d and p and $d=\ell m$, $p=\ell n$ then $C_{2p}(d,p)$ is isomorphic to ℓ copies of $G_{2n}(1,n)$ if m is odd or of $C_{2n}(2,n)$ if m is even.*

Theorem 1 ([6]) *The circulant graph $G=C_{2p}(d,p)$ is type 1 if and only if m is even and G is not isomorphic to $\ell C_{10}(2,5)$, otherwise G is type 2.*

Consequently every cubic circulant $C_{2p}(1,p)$ along with $C_{10}(2,5)$ are type 2.

Theorem 2 ([6]) *For every type 2 cubic circulant graph G, we have $\chi_c''(G)\leq\frac{9}{2}$.*

Lemma 4 *The fractional total chromatic number of a cubic circulant graph $G=C_{2p}(1,p)$ having a fractional (η,μ_1,μ_2)-stable with $\eta\geq\max(\mu_1,2\mu_2)$, satisfies*

$$\chi_f''(C_{2p}(1,p))\leq\frac{2(p-\mu_1-\mu_2)}{\eta}+3.$$

Proof : By using the result obtained in Lemma 1 we have

$$\begin{aligned}\chi_f''(G) &\leq \tfrac{2p}{\eta}+(\tfrac{\eta-\mu_1}{\eta})\chi_f'(C_{\frac{2p}{\gcd(2p,1)}})-\tfrac{2\mu_2}{\eta}+1 \quad\Rightarrow\\ \chi_f''(G) &\leq \tfrac{2p}{\eta}+(\tfrac{\eta-\mu_1}{\eta})\chi_f'(C_{2p})-\tfrac{2\mu_2}{\eta}+1.\end{aligned}$$

As $\chi_f^{'}(C_{2p}) = 2$, we obtain

$$\chi_f^{''}(G) \leq \frac{2p + 2(\eta - \mu_1) - 2\mu_2}{\eta} + 1 = \frac{2(p - \mu_1 - \mu_2)}{\eta} + 3. \square$$

$\square$

If the graph G possesses a fractional $(a, a, 0)$-stable, we can obtain the next corollary by applying Lemma 4.

Corollary 2 *For every cubic circulant graph $G = C_{2p}(1, p)$ possessing a fractional $(a, a, 0)$-stable, the fractional total chromatic number satisfies*

$$\chi_f^{''}(G) \leq \frac{2p}{a} + 1.$$

Therefore, in order to bound the fractional total chromatic number of cubic circulant graphs, we now have to find fractional balanced stables or fractional $(a, a, 0)$-stables which are the most complete possible.

Lemma 5 *For any $q \geq 1$, the circulant graph $G = C_{8q}(1, 4q)$ has a fractional $(2q, 2q, q)$-stable.*

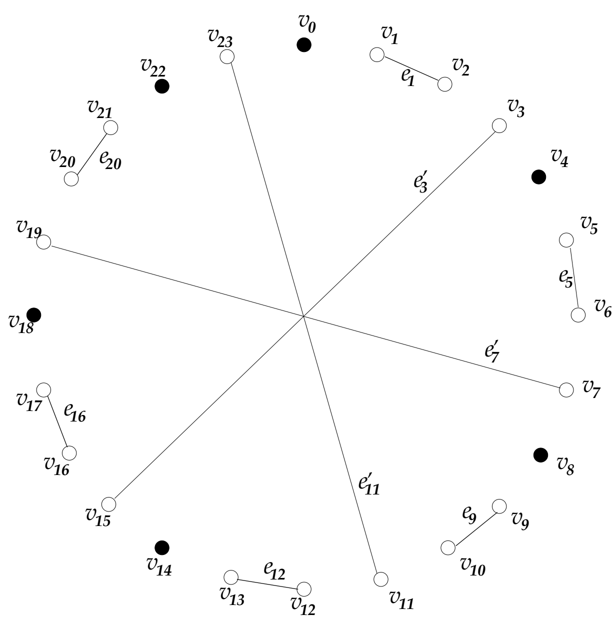

Figure 2: A $(6, 6, 3)$-stable in $C_{24}(1, 12)$.

Proof : In fact in this case, the fractional stable is reduced to a single $(2q, 2q, q)$-stable S defined as follows : for $0 \leq j \leq q - 1$,

$$\begin{cases} v_{4j}, v_{4q+4j+2} \in V(S) \\ e^1_{4j+1}, e^1_{4q+4j} \in E_1(S) \\ e^2_{4j+3} \in E_2(S). \end{cases}$$

Such a stable is presented in Figure 2 for $q = 3$.

It is routine to show that S is an independent set.

Therefore, S is a (η, μ_1, μ_2)-stable with $\eta = |V(S)| = 2q$, $\mu_1 = |E_1(S)| = 2q$ and $\mu_2 = |E_2(S)| = q$ which is also a fractional $(2q, 2q, q)$-stable. $\square$ $\square$

Lemma 6 *For any $q \geq 1$ and $1 \leq r \leq 3$, the cubic circulant graph $G = C_{8q+2r}(1, 4q+r)$ has a fractional $(a, a, 0)$-stable, where the value of a is given in the following table:*

	$q \equiv 0 \bmod 3$	$q \equiv 1 \bmod 3$	$q \equiv 2 \bmod 3$
$r = 1$	$a = \frac{16q+3}{6}$	$a = \frac{8q+1}{3}$	$a = \frac{16q+1}{6}$
$r = 2$	$a = \frac{8q+3}{3}$	$a = \frac{16q+5}{6}$	$a = \frac{16q+7}{6}$
$r = 3$	$a = \frac{16q+9}{6}$	$a = \frac{16q+11}{6}$	$a = \frac{8q+5}{3}$

Proof : **Case 1:** $n = 8q+2$**.** A fractional $(a, a, 0)$-stable of $G = C_{8q+2}(1, 4q+1)$ can be obtained from the two stables S_1 and S_2 defined below, except in the case $q \equiv 1 \bmod 3$ where the fractional stable is reduced to a single $(a, a, 0)$-stable S_1:

- If $q \equiv 0 \bmod 3$, let S_1 and S_2 be the two stables defined by

$$\begin{cases} v_{3j} \in V(S_1) & 0 \leq j \leq \frac{8q}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{8q}{3} - 1. \end{cases}$$

$$\begin{cases} v_{3j+2} \in V(S_2) & 0 \leq j \leq \frac{8q}{3} - 1, \\ e^1_{3j} \in E_1(S_2) & 0 \leq j \leq \frac{8q}{3}. \end{cases}$$

  Thus, we obtain $n(S_1) = m_1(S_2) = \frac{8q+3}{3}$, $m_1(S_1) = n(S_2) = \frac{8q}{3}$ and $m_2(S_1) = m_2(S_2) = 0$.

  Therefore, S_1 and S_2 form a fractional $(\frac{16q+3}{6}, \frac{16q+3}{6}, 0)$-stable.

- If $q \equiv 1 \bmod 3$, then for $0 \leq j \leq \frac{8q-2}{3}$,

$$\begin{cases} v_{3j} \in V(S_1), \\ e^1_{3j+1} \in E_1(S_1). \end{cases}$$

  Therefore, S_1 is a $(\frac{8q+1}{3}, \frac{8q+1}{3}, 0)$-stable.

- If $q \equiv 2 \bmod 3$ then

$$\begin{cases} v_{8q} \in V(S_1), & & \\ v_{3j} \in V(S_1), & e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{4q-2}{3}, \\ v_{4q+3j+3} \in V(S_1), & e^1_{4q+3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{4q-5}{3}. \end{cases}$$

$$\begin{cases} e^1_{8q} \in E_1(S_2) & & \\ v_{3j+2} \in V(S_2) & e^1_{3j} \in E_1(S_2) & 0 \leq j \leq \frac{4q-2}{3}, \\ v_{4q+3j+4} \in V(S_2) & e^1_{4q+3j+2} \in E_1(S_2) & 0 \leq j \leq \frac{4q-5}{3}. \end{cases}$$

  Thus, we obtain $n(S_1) = m_1(S_2) = \frac{8q+2}{3}$, $m_1(S_1) = n(S_2) = \frac{8q-1}{3}$ and $m_2(S_1) = m_2(S_2) = 0$.

  Therefore, S_1 and S_2 form a fractional $(\frac{16q+1}{6}, \frac{16q+1}{6}, 0)$-stable.

Case 2 : $n = 8q+4$**.** The proof is similar with the one of Case 1 : a fractional $(a, a, 0)$-stable of $C_{8q+4}(1, 4q+2)$ can be obtained from the two stables S_1 and S_2 except in the case $q \equiv 0 \bmod 3$ where the fractional stable is reduced to a single $(a, a, 0)$-stable S_1:

	S_1	S_2	a
$q \equiv 0 \bmod 3$	$\begin{cases} v_{3j} \in V(S_1) & 0 \leq j \leq \frac{8q}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{8q}{3}. \end{cases}$		$\frac{8q+3}{3}$
$q \equiv 1 \bmod 3$	$\begin{cases} v_{8q+2} \in V(S_1) & \\ v_{3j} \in V(S_1) & 0 \leq j \leq \frac{4q-1}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{4q-1}{3}, \\ v_{4q+3j+4} \in V(S_1) & 0 \leq j \leq \frac{4q-4}{3}, \\ e^1_{4q+3j+2} \in E_1(S_1) & 0 \leq j \leq \frac{4q-4}{3}. \end{cases}$	$\begin{cases} e^1_{8q+2} \in E_1(S_2) & \\ v_{3j+2} \in V(S_2) & 0 \leq j \leq \frac{4q-1}{3}, \\ e^1_{3j} \in E_1(S_2) & 0 \leq j \leq \frac{4q-1}{3}, \\ v_{4q+3j+3} \in V(S_2) & 0 \leq j \leq \frac{4q-4}{3}, \\ e^1_{4q+3j+4} \in E_1(S_2) & 0 \leq j \leq \frac{4q-4}{3}. \end{cases}$	$\frac{16q+5}{6}$
$q \equiv 2 \bmod 3$	$\begin{cases} v_{3j} \in V(S_1) & 0 \leq j \leq \frac{8q+2}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{8q-1}{3}. \end{cases}$	$\begin{cases} v_{3j+2} \in V(S_2) & 0 \leq j \leq \frac{8q-1}{3}, \\ e^1_{3j} \in E_1(S_2) & 0 \leq j \leq \frac{8q+2}{3}. \end{cases}$	$\frac{16q+7}{6}$

Case 3 : $n = 8q + 6$**.** Similarly, a fractional $(a, a, 0)$-stable of $C_{8q+6}(1, 4q + 3)$ can be obtained from the two stables S_1 and S_2 except in the case $q \equiv 2 \bmod 3$ where the fractional stable is reduced to a single $(a, a, 0)$-stable :

	S_1	S_2	a
$q \equiv 0 \bmod 3$	$\begin{cases} v_{8q+4} \in V(S_1) & \\ v_{3j} \in V(S_1) & 0 \leq j \leq \frac{4q}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{4q}{3}, \\ v_{4q+3j+5} \in V(S_1) & 0 \leq j \leq \frac{4q-3}{3}, \\ e^1_{4q+3j+3} \in E_1(S_1) & 0 \leq j \leq \frac{4q-3}{3}. \end{cases}$	$\begin{cases} e^1_{8q+4} \in V(S_2) & \\ v_{3j+2} \in V(S_2) & 0 \leq j \leq \frac{4q}{3}, \\ e^1_{3j} \in E_1(S_2) & 0 \leq j \leq \frac{4q}{3}, \\ v_{4q+3j+4} \in V(S_2) & 0 \leq j \leq \frac{4q-3}{3}. \\ e^1_{4q+3j+5} \in E_1(S_2) & 0 \leq j \leq \frac{4q-3}{3}. \end{cases}$	$\frac{16q+9}{6}$
$q \equiv 1 \bmod 3$	$\begin{cases} v_{3j} \in V(S_1) & 0 \leq j \leq \frac{8q+4}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{8q+1}{3}. \end{cases}$	$\begin{cases} v_{3j+2} \in V(S_2) & 0 \leq j \leq \frac{8q+1}{3}, \\ e^1_{3j} \in E_1(S_2) & 0 \leq j \leq \frac{8q+4}{3}. \end{cases}$	$\frac{16q+11}{6}$
$q \equiv 2 \bmod 3$	$\begin{cases} v_{3j} \in V(S_1) & 0 \leq j \leq \frac{8q+2}{3}, \\ e^1_{3j+1} \in E_1(S_1) & 0 \leq j \leq \frac{8q+2}{3}. \end{cases}$		$\frac{8q+5}{3}$

In each case, it is straightforward to verify that the defined sets are independent sets. □□ We obtain the next corollary by combining Lemma 4, Corollary 2, Lemma 5 and Lemma 6.

Corollary 3 *Let $G = C_{2p}(1, p)$ be a circulant graph with $p = 4q + r$ for $0 \leq r \leq 3$ and $q \geq 1$. The fractional total chromatic number of G satisfies*

if $r = 0$ then

$$\chi''_f(G) = 4,$$

if $r = 1$ then

$$\chi''_f(G) \leq \begin{cases} 4 + \frac{3}{16q+3} & \text{if } q \equiv 0 \bmod 3, \\ 4 + \frac{3}{8q+1} & \text{if } q \equiv 1 \bmod 3, \\ 4 + \frac{9}{16q+1} & \text{if } q \equiv 2 \bmod 3, \end{cases}$$

if $r = 2$ then

$$\chi''_f(G) \leq \begin{cases} 4 + \frac{3}{8q+3} & \text{if } q \equiv 0 \bmod 3, \\ 4 + \frac{9}{16q+5} & \text{if } q \equiv 1 \bmod 3, \\ 4 + \frac{3}{16q+7} & \text{if } q \equiv 2 \bmod 3, \end{cases}$$

if $r = 3$ then

$$\chi''_f(G) \leq \begin{cases} 4 + \frac{9}{16q+9} & \text{if } q \equiv 0 \bmod 3, \\ 4 + \frac{3}{16q+11} & \text{if } q \equiv 1 \bmod 3, \\ 4 + \frac{3}{8q+5} & \text{if } q \equiv 2 \bmod 3. \end{cases}$$

In a more reduced way, we obtain the following bound which is asymptotically optimal :

Corollary 4 *For any circulant graph $G = C_{2p}(1, p)$, then*

$$\begin{cases} \chi''_f(G) = 4 & \text{if } p \equiv 0 \bmod 4, \\ \chi''_f(G) \leq 4 + \frac{9}{4p-3} & \text{otherwise.} \end{cases}$$

To be complete with cubic circulant graphs, we have to treat the case of $C_{10}(2, 5)$ which is type 2: we have $\chi''_f(C_{10}(2, 5)) \leq \chi''_c(C_{10}(2, 5)) = \frac{13}{3}$ by the result of [6]. Indeed, by a case analysis, we can easily be convinced that there does not exist a complete balanced fractional stable (and thus $\chi''_f(C_{10}(2, 5)) > 4$. Moreover, it seems that $\frac{13}{3}$ is the exact value of the fractional total chromatic index of $C_{10}(2, 5)$, but we did not go further to prove it.

4 4-regular circulant graphs

In this section we study the total chromatic number and fractional total chromatic number for 4-regular circulant graphs. In the first we give the total chromatic number for some 4-regular circulant graphs.

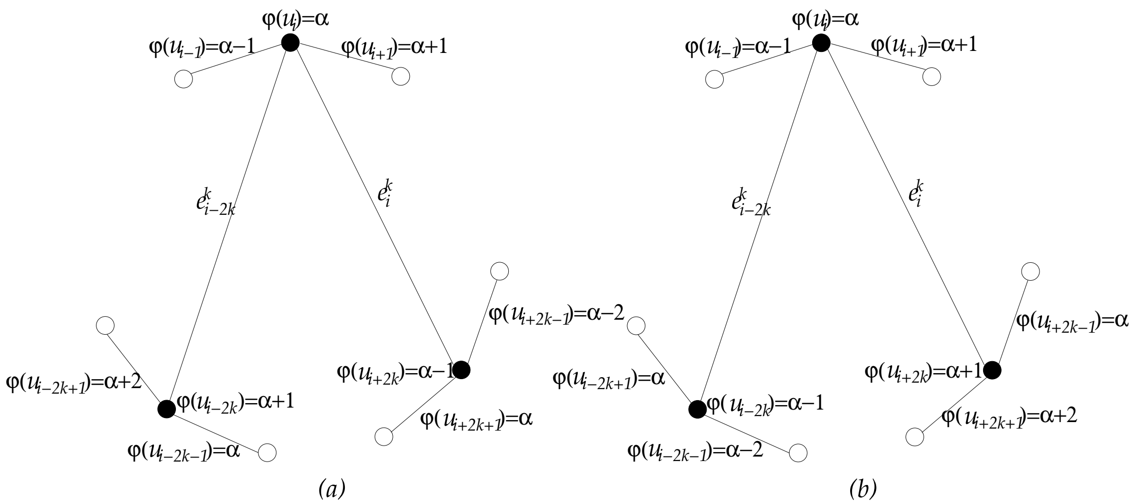

Figure 3: Total colouring of $C_{5p}(1,k)$: (a) $k \equiv 2 \bmod 5$, (b) $k \equiv 3 \bmod 5$.

Theorem 3 *Every 4-regular circulant graph $G = C_{5p}(1,k)$ is type 1 for any positive integers p and $k < \frac{5p}{2}$ with $k \equiv 2 \bmod 5$ or $k \equiv 3 \bmod 5$.*

Proof: We rename the elements of $V(G) \cup E_1(G)$ as follows : $u_0 = v_0, u_1 = e^1_0, u_2 = v_1, \ldots, u_{10p-2} = v_{5p-1}, u_{10p-1} = e^1_{5p-1}$.

We define a total colouring function φ by

$$\varphi(u_i) = i \bmod 5, \quad 0 \leq i \leq 10p-1.$$

In order to colour the internal edges of G, we consider two cases (the indices on the u_i are modulo $10p$ and the colours are modulo 5):

Case 1 : $k \equiv 2 \bmod 5$. Suppose that the element u_i is a vertex and $\varphi(u_i) = \alpha$. Then, by definition of φ, we have (modulo 5) $\varphi(u_{i-1}) = \alpha - 1$ and $\varphi(u_{i+1}) = \alpha + 1$.

The two neighboring vertices of u_i linked by an edge of length k are u_{i-2k} and u_{i+2k}, and by definition of φ, we have $\varphi(u_{i-2k-1}) = \alpha$, $\varphi(u_{i-2k}) = \alpha+1$, $\varphi(u_{i-2k+1}) = \alpha+2$, $\varphi(u_{i+2k-1}) = \alpha-2$, $\varphi(u_{i+2k}) = \alpha - 1$ and $\varphi(u_{i+2k+1}) = \alpha$, as Figure 3 (a) illustrates.

To colour the two internal edges $e^k_{i-2k} = (u_{i-2k}, u_i)$ and $e^k_i = (u_i, u_{i+2k})$ we must have $\varphi(e^k_{i-2k}) \notin \{\varphi(u_{i-1}), \varphi(u_i), \varphi(u_{i+1}), \varphi(u_{i-2k-1}), \varphi(u_{i-2k}), \varphi(u_{i-2k+1})\} = \{\alpha-1, \alpha, \alpha+1, \alpha+2\}$ and $\varphi(e^k_i) \notin \{\varphi(u_{i-1}), \varphi(u_i), \varphi(u_{i+1}), \varphi(u_{i+2k-1}), \varphi(u_{i+2k}), \varphi(u_{i+2k+1}) = \{\alpha-2, \alpha-1, \alpha, \alpha+1\}$. Thus, we can set $\varphi(e^k_{i-2k}) = \alpha+3$ and $\varphi(e^k_i) = \alpha+2$.

Therefore we have used five colours (notice that $\alpha-2 \equiv (\alpha+3) \bmod 5$), consequently $\chi''(G_{5p}(1,k)) = 5$.

Case 2 : $k \equiv 3 \bmod 5$. As for the precedent case, suppose that the element u_i is a vertex and $\varphi(u_i) = \alpha$. Then, by definition, $\varphi(u_{i-1}) = \alpha - 1$ and $\varphi(u_{i+1}) = \alpha + 1$.

The two neighboring vertices linked by an edge of length k are u_{i-2k} and u_{i+2k}, and by definition of φ, we have $\varphi(u_{i-2k-1}) = \alpha - 2$, $\varphi(u_{i-2k}) = \alpha - 1$, $\varphi(u_{i-2k+1}) = \alpha$, $\varphi(u_{i+2k-1}) = \alpha$, $\varphi(u_{i+2k}) = \alpha + 1$ and $\varphi(u_{i+2k+1}) = \alpha + 2$, as Figure 3 (b) illustrates.

To colour the two internal edges $e^k_{i-2k} = (u_{i-2k}, u_i)$ and $e^k_i = (u_i, u_{i+2k})$ we must have $\varphi(e^k_{i-2k}) \notin \{\varphi(u_{i-1}), \varphi(u_i), \varphi(u_{i+1}), \varphi(u_{i-2k-1}), \varphi(u_{i-2k}), \varphi(u_{i-2k+1})\} = \{\alpha-2, \alpha-1, \alpha, \alpha+1\}$ and $\varphi(e^k_i) \notin \{\varphi(u_{i-1}), \varphi(u_i), \varphi(u_{i+1}), \varphi(u_{i+2k-1}), \varphi(u_{i+2k}), \varphi(u_{i+2k+1}) = \{\alpha-1, \alpha, \alpha+1, \alpha+2\}$. Thus, we can set $\varphi(e^k_{i-2k}) = \alpha+2$ and $\varphi(e^k_i) = \alpha+3$.

Therefore we have used five colours, consequently $\chi''(C_{5p}(1,k)) = 5$. □ □

Theorem 4 *Every 4-regular circulant graph $G = C_{6p}(1,k)$ is type 1 for any positive integer $p \geq 3$ and $k < 3p$ with $k \equiv 1 \bmod 3$ or $k \equiv 2 \bmod 3$.*

Proof :Let $G = C_{6p}(1,k)$ and let $q = \gcd(6p,k)$. As $k \not\equiv 0 \bmod 3$, then 3 divides $\frac{6p}{q}$.

Let C_i be the cycle in G induced by the edges of length k that contains the vertex v_i, $0 \leq i \leq q-1$. Notice that if $6p$ and k are relatively prime, then there is only one cycle C_0. Let $C_0 = (u_0, \ldots, u_{\frac{6p}{q}} - 1)$, with $u_{2j} = v_{jk \bmod 6p}$ and $u_{2j+1} = e^k_{jk \bmod 6p}$, $0 \leq j \leq \frac{6p}{q} - 1$.

Colour vertices and edges of C_0 with colours $0, 1, 2$ cyclically, i.e. set $\varphi(u_i) = i \bmod 3$. The other cycles C_i, $1 \leq i \leq q-1$, are coloured similarly by setting $\varphi(C_{2i+1}) = \varphi(C_{2i}) + 1 \bmod 3$ and $\varphi(C_{2i+2}) = \varphi(C_{2i})$, $0 \leq i \leq \frac{q-3}{2}$, and $\varphi(C_{q-1}) = \varphi(C_0) + 2$ if q is even. That is if $v_j \in C_0$ for some j, then $\varphi(v_{j+2i+1}) = (\varphi(v_j) + 1) \bmod 3$, $\varphi(v_{j+2i+2}) = \varphi(v_j)$, $0 \leq i \leq \frac{q-3}{2}$ and $\varphi(v_{j+q-1}) = (\varphi(v_j) + 2) \bmod 3$ if q is even.

For the edges of length one, set $\varphi(e^1_i) = i \bmod 3 + 2$, $0 \leq i \leq 6p - 1$.

Clearly, by definition, φ colours properly the edges of G. Now, let us show that the colouring of the vertices is also proper: two adjacent vertices of C_i and C_{i+1} have not the same colour by definition. Then, it just remains to see that the vertex v_{q-1} of C_{q-1} has a different colour of that of vertex v_q of C_0: We have $\varphi(v_{q-1}) = \varphi(v_0) = 0$ if q is odd or $\varphi(v_{q-1}) = \varphi(v_0) + 2 = 2$ if q is even and $\varphi(v_q) = 1$. Thus φ is a proper total colouring of G with five colours; hence G is type 1. □ □

Remark that not all circulant graphs $G = C_n(1,3)$ are type 1: for instance $C_8(1,3)$ which is isomorphic to $K_{4,4}$ is type 2 since $\alpha''(C_8(1,3)) \leq 4$ thus (by properties 1 and 2) $\chi''(C_8(1,3)) \geq \frac{24}{4} = 6$.

Nevertheless, it seems that most of the 4-regular circulant graphs are of type 1, but we are not able to prove it. We then now try to determine the fractional chromatic number of 4-regular circulant graphs.

Lemma 7 *Let n, q and k be integers, $k \not\equiv 0 \bmod 3$, $2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ and $2q \leq n$ if $q \equiv 0 \bmod k$, $q + k(\lfloor \frac{q}{k} \rfloor + 1) \leq n$ otherwise. If $n - 2q \equiv 0 \bmod 3$ then the circulant graph $G = C_n(1,k)$ possesses a complete (a,a,q)-stable.*

Proof :We prove this lemma by constructing the (a,a,q)-stable. Starting from v_0 and going in the increasing direction of the vertex indices, we first select q independent edges of length $k \not\equiv 0 \bmod 3$ successively in $C_n(1,k)$. Suppose that $q = \alpha k + \beta$, $0 \leq \beta \leq k-1$.

- If $\beta = 0$, the q independent edges of length k use $2q$ consecutive vertices. Therefore we must have $2q \leq n$ and it remains $n - 2q$ vertices.
- If $1 \leq \beta \leq k-1$, q independent edges of length k use $2\alpha k + \beta + k$ vertices. Therefore we must have $2\alpha k + k + \beta = q + k(\lfloor \frac{q}{k} \rfloor + 1) \leq n$ and it remains a block A of $k - \beta$ consecutive vertices and a block B of $n - (2\alpha k + k + \beta)$ consecutive vertices. Thus a total of $n - 2q$ vertices.

In the first case ($\beta = 0$), as $k \not\equiv 0 \bmod 3$, an independent set with a vertices and a edges of length 1 can be constructed by selecting alternatively a vertex and an edge along the consecutive remaining vertices.

Similarly, in the second case ($1 \leq \beta \leq k-1$), the independent set of a vertices and a edges of length 1 can be constructed by alternating the vertices and the edges of length 1 along the remaining vertices that are divided in two blocks A and B, using the following rule : if the first selected element in block A is a vertex, then the first selected element in block B must be an edge (since the first vertex of B is at distance k with the one of A) and conversely. An illustration is given in Figure 4 for the cases $|A| \equiv 1 \bmod 3$, $|B| \equiv 2 \bmod 3$ and $|A| \equiv 2 \bmod 3$, $|B| \equiv 1 \bmod 3$.

If $q < k$ there could be an edge of length k between a vertex at the beginning of block A and a vertex at the end of block B. If $|A| \equiv 1 \bmod 3$ and $|B| \equiv 2 \bmod 3$ or $|A| \equiv 2 \bmod 3$ and $|B| \equiv 1 \bmod 3$, this is not the case since by definition of the stable, the pattern is symmetrical in each block, i.e. if the first selected element of the block is a vertex (an edge, respectively) then the last one is a vertex (an edge, respectively) too. If $|A| \equiv 0 \bmod 3$ and $|B| \equiv 0 \bmod 3$, then the stable is defined by $\{e^k_0, e^k_1, \ldots, e^k_{q-1}\} \cup \{v_{q+3j}, e^1_{q+3j+1} : 0 \leq j \leq \frac{k-q}{3} - 1\} \cup \{v_{n-3i-1}, e^1_{n-3i-3}, : 0 \leq i \leq \frac{n-k-q}{3} - 1\}$. Thus the only possibility to have an edge between two vertices of this

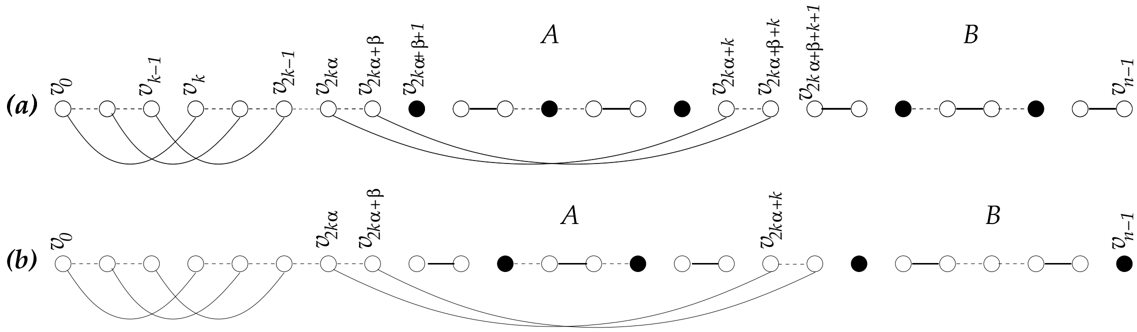

Figure 4: Structure of the (a, a, q)-stable when $|A| \equiv 1 \bmod 3$ and $|B| \equiv 2 \bmod 3$ (a); and when $|A| \equiv 2 \bmod 3$ and $|B| \equiv 1 \bmod 3$ (b).

set is if $k - 3i - 1 = q + 3j$ for some i, j i.e. if $3(i + j) = k - q - 1$, which is impossible since $k - q = |A| \equiv 0 \bmod 3$.

By hypothesis, $n - 2q \equiv 0 \bmod 3$ and thus $a = \frac{n-2q}{3}$. Hence, the (a, a, q)-stable is complete. □ □

Lemma 8 *For every positive integers $n \geq 9$ and $k \not\equiv 0 \bmod 3$, $2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, the circulant graph $C_n(1, k)$ possesses a fractional $(\frac{n}{5}, \frac{n}{5}, \frac{n}{5})$-stable.*

Proof :Let $G = C_n(1, k)$. By Lemma 7, we can define one or two complete stables S_1 and S_2 in the graph G as described in the following table :

	S_1	S_2
$n = 5p, p \geq 2$	(p, p, p)-stable	
$n = 5p + 1, p \geq 2$	$(p + 1, p + 1, p - 1)$-stable	$(p - 1, p - 1, p + 2)$-stable
$n = 5p + 2, p \geq 2$	$(p + 2, p + 2, p - 2)$-stable	$(p, p, p + 1)$-stable
$n = 5p + 3, p \geq 2$	$(p - 1, p - 1, p + 3)$-stable	$(p + 1, p + 1, p)$-stable
$n = 5p + 4, p \geq 1$	$(p, p, p + 2)$-stable	$(p + 2, p + 2, p - 1)$-stable

In each case one can see that $n - 2m_2(S_1) \equiv 0 \bmod 3$ and $n - 2m_2(S_2) \equiv 0 \bmod 3$, thus the condition of Lemma 7 is satisfied.

Moreover, for each of the above (a, a, q)-stable, it can be shown that the condition $q + k(\lfloor \frac{q}{k} \rfloor + 1) \leq n$ is verified. We just prove it for the most "critical" case: the $(p-1, p-1, p+3)$-stable when $n = 5p + 2$ (the other cases can be proved in a similar way).

If $k < q = p + 3$, then we have $k(\lfloor \frac{q}{k} \rfloor + 1) \leq k(\frac{p+2}{k} + 1) = p + 2 + k \leq 2p + 4$ and thus we have to show that $p + 3 + 2p + 4 \leq n = 5p + 2$ which is verified whenever $p \geq 2$.

If $k > q = p + 3$, then $k(\lfloor \frac{q}{k} \rfloor + 1) = k$. Hence we need $p + 3 + k \leq 5p + 2$, i.e. $k \leq 4p$ which is verified since by hypothesis $k \leq \lfloor \frac{n-1}{2} \rfloor$.

Now, a fractional $(\frac{n}{5}, \frac{n}{5}, \frac{n}{5})$-stable is obtained from 3 stables S_1 and 2 stables S_2 when $n = 5p+1$, or $5p + 4$, and from one stable S_1 and 4 stables S_2 when $n = 5p + 2$ or $5p + 3$ (in the case $n = 5p$, the fractional stable is reduced to a single (p, p, p)-stable). □ □

For the case $k = 3$, the next lemma gives a similar result, using an ad-hoc method.

Lemma 9 *For any positive integer $n \geq 9$, $n \neq \{12, 13, 17\}$, the 4-regular circulant graph $G = C_n(1, 3)$ has a fractional $(\frac{n}{5}, \frac{n}{5}, \frac{n}{5})$-stable.*

Proof :To prove this lemma, we construct for each value of the residue of n modulo 5, the one or two stables that form the fractional $(\frac{n}{5}, \frac{n}{5}, \frac{n}{5})$-stable :

Case 1 : $n = 5p$**.** We define the (p, p, p)-stable S as the table shows:

$S : (p, p, p)$-stable
$\left\{ \begin{array}{ll} v_{5i} \in V(S_1) & 0 \leq i \leq p-1, \\ e^1_{2+5i} \in E_1(S_1) & 0 \leq i \leq p-1. \\ e^3_{1+5i} \in E_3(S_1) & 0 \leq i \leq p-1, \end{array} \right.$

Case 2 : $n = 5p + 1$ and $p \geq 2$. We define two stables S_1 and S_2 as follows:

$S_1 : (p+1, p-1, p+1)$-stable	$S_2 : (p-1, p+2, p-1)$-stable
$\begin{cases} v_1, v_6, v_{10} \in V(S_1) & \\ e^1_8 \in E_1(S_1) & \\ e^3_0, e^3_2, e^3_4 \in E_3(S_1) & \\ v_{15+5i} \in V(S_1) & 0 \leq i \leq p-3, \\ e^1_{12+5i} \in E_1(S_1) & 0 \leq i \leq p-3, \\ e^3_{11+5i} \in E_3(S_1) & 0 \leq i \leq p-3. \end{cases}$	$\begin{cases} e^1_0, e^1_2, e^1_4 \in E_1(S_2) & \\ v_{6+5i} \in V(S_2) & 0 \leq i \leq p-2, \\ e^1_{8+5i} \in E_1(S_2) & 0 \leq i \leq p-2, \\ e^3_{7+5i} \in E_3(S_2) & 0 \leq i \leq p-2. \end{cases}$

Therefore, we obtain a fractional $(\frac{5p+1}{5}, \frac{5p+1}{5}, \frac{5p+1}{5})$-stable from three stables S_1 and two stables S_2.

Case 3 : $n = 5p + 2$ and $p > 3$. We define two stables S_1 and S_2 as follows:

$S_1 : (p, p+1, p)$-stable	$S_2 : (p+2, p-2, p+2)$-stable
$\begin{cases} e^1_0 \in E_1(S_1) & \\ v_{6+5i} \in V(S_1) & 0 \leq i \leq p-1, \\ e^1_{3+5i} \in E_1(S_1) & 0 \leq i \leq p-1, \\ e^3_{2+5i} \in E_3(S_1) & 0 \leq i \leq p-1. \end{cases}$	$\begin{cases} v_0, v_2, v_7, v_{11}, v_{13}, v_{18} \in V(S_2) & \\ e^1_9, e^1_{20} \in E_1(S_2) & \\ e^3_1, e^3_3, e^3_5, e^3_{12}, e^3_{14}, e^3_{16} \in E_3(S_2) & \\ v_{22+5i} \in V(S_2) & 0 \leq i \leq p-5, \\ e^1_{24+5i} \in E_1(S_2) & 0 \leq i \leq p-5, \\ e^3_{23+5i} \in E_3(S_2) & 0 \leq i \leq p-5. \end{cases}$

Thus, we obtain a fractional $(\frac{5p+2}{5}, \frac{5p+2}{5}, \frac{5p+2}{5})$-stable from four stables S_1 and one stable S_2.

Case 4 : $n = 5p + 3$ and $p > 2$. We define two stables S_1 and S_2 as follows:

$S_1 : (p+1, p, p+1)$-stable	$S_2 : (p-1, p+3, p-1)$-stable
$\begin{cases} v_1, v_6, v_{10}, v_{15} \in V(S_1) & \\ e^1_8, e^1_{12}, e^1_{16} \in E_1(S_1) & \\ e^3_0, e^3_2, e^3_4, e^3_{11} \in E_3(S_1) & \\ v_{22+5i} \in V(S_1) & 0 \leq i \leq p-4, \\ e^1_{19+5i} \in E_1(S_1) & 0 \leq i \leq p-4, \\ e^3_{18+5i} \in E_3(S_1) & 0 \leq i \leq p-4. \end{cases}$	$\begin{cases} e^1_0, e^1_2, e^1_4, e^1_6 \in E_1(S_2) & \\ v_{8+5i} \in V(S_2) & 0 \leq i \leq p-2, \\ e^1_{10+5i} \in E_1(S_2) & 0 \leq i \leq p-2, \\ e^3_{9+5i} \in E_3(S_2) & 0 \leq i \leq p-2. \end{cases}$

Therefore, we obtain a fractional $(\frac{5p+3}{5}, \frac{5p+3}{5}, \frac{5p+3}{5})$-stable from four stables S_1 and one stable S_2.

Case 5 : $n = 5p + 4$ and $p \geq 1$. As for the previous cases we define the two stables which form the fractional $(\frac{5p+4}{5}, \frac{5p+4}{5}, \frac{5p+4}{5})$-stable as follows:

$S_1 : (p, p+2, p)$-stable	$S_2 : (p+2, p-1, p+2)$-stable
$\begin{cases} e^1_0, e^1_2 \in E_1(S_1) & \\ v_{4+5i} \in V(S_1) & 0 \leq i \leq p-1, \\ e^1_{6+5i} \in E_1(S_1) & 0 \leq i \leq p-1, \\ e^3_{5+5i} \in E_3(S_1) & 0 \leq i \leq p-1. \end{cases}$	$\begin{cases} v_0, v_2, v_7 \in V(S_2) & \\ e^3_1, e^3_3, e^3_5 \in E_3(S_2) & \\ v_{9+5i} \in V(S_2) & 0 \leq i \leq p-2, \\ e^1_{11+5i} \in E_1(S_2) & 0 \leq i \leq p-2, \\ e^3_{10+5i} \in E_3(S_2) & 0 \leq i \leq p-2. \end{cases}$

Thus, we obtain a fractional $(\frac{5p+4}{5}, \frac{5p+4}{5}, \frac{5p+4}{5})$-stable from three stables S_1 and two stables S_2. $\square$ $\square$

Applying Lemma 8, Lemma 9 and Corollary 1 (or Lemma 2), we obtain the following two corollaries :

Corollary 5 *For any circulant graph $C_n(1,3)$ with $n \geq 9$ and $n \notin \{12, 13, 17\}$, the fractional total chromatic number is*

$$\chi''_f(C_n(1,3)) = 5.$$

Corollary 6 *For any circulant graph $C_n(1,k)$ with $n \geq 9$, $k \leq \lfloor \frac{n-1}{2} \rfloor$ and $k \not\equiv 0 \bmod 3$, the fractional total chromatic number is*

$$\chi''_f(C_n(1,k)) = 5.$$

5 Concluding remarks

For cubic type 2 circulant graphs, we have derived asymptotically optimal upper bounds on the fractional total chromatic number. An interesting task could be to find good approximate of the circular total chromatic number of these graphs. In particular, is the $\frac{9}{2}$ bound of Hackmann and Kemnitz close to the optimal? In a more general setting, like for the non-total version for which some circulants are known to be *star extremal* (i.e. satisfy $\chi_f = \chi_c$, see [10]), are there circulant graphs for which the fractional total chromatic number is equal to the circular total chromatic number?

For 4-regular circulant graphs $C_n(1,k)$, we have determined the total chromatic number for some specific cases and the fractional chromatic number when $k \not\equiv 0 \bmod 3$. For $k \equiv 0 \bmod 3$, our method to construct a (a,a,q)-stable does not work because it is not possible to build the stable by alternating vertices and edges of length one too many times. Nevertheless, it seems possible to construct complete balanced fractional stables in this case; hence we think that $\chi''_f(C_n(1,3t)) = 5$. Moreover, having computed (with the help of the computer) the total chromatic number of $C_n(1,k)$ for the first values of n and k, we conjecture that all but except a finite number of 4-regular circulant graphs $C_n(1,k)$ are type 1.

Finally, our method to compute the fractional total chromatic number of 4-regular circulant graphs $C_n(1,k)$, $k \not\equiv 0 \bmod 3$ could perhaps be used to compute the fractional total chromatic number of ℓ-regular circulant graphs $C_n(1,k_1,k_2,\ldots,k_{\ell-1})$ with $k_i \not\equiv 0 \bmod 3$, $1 \le i \le \ell-1$.

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