On a vertex-edge marking game on graphs

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Abstract

The study of a variation of the marking game, in which the first player marks vertices and the second player marks edges of an undirected graph was proposed by Bartnicki et al. in [Game chromatic number of Cartesian product graphs, Electron. J. Combin. 15 (2008) #R72]. In this game, the goal of the second player is to mark as many edges around an unmarked vertex as possible, while the first player wants just the opposite. In this paper, we prove various bounds for the corresponding graph invariant, the vertex-edge coloring number $\mathrm{col}_{ve}(G)$ of a graph G. In particular, every (finite or infinite) graph G whose edges can be oriented in such a way that the maximum out-degree is bounded by an integer d has $\mathrm{col}_{ve}(G) \leq d+2$. We investigate this invariant in (classes of) planar graphs, including some infinite lattices. We present a close connection between the vertex-edge coloring number of a graph G and the game coloring number of the subdivision graph $S(G)$. In our main result, we bound the vertex-edge coloring number in complete graphs from below and from above, and while $\mathrm{col}_{ve}(K_n) \leq \lceil \log_2 n \rceil + 2$, the difference between the upper and the lower bound is roughly $\log_2(\log_2 n)$. The latter results are in fact true for any multigraph whose underlying graph is K_n.

Keywords: marking game, coloring game, degenerate graph, complete graph

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1 Introduction

The coloring game on graphs was introduced independently by Gardner [11] and Bodlander [4], and was henceforth studied by a number of authors. The initial version of the coloring game triggered numerous investigations, which resulted in the development of various methods and strategies; see the brief survey on different kinds of coloring games by Bartnicki et al. [3]. A close variation of the coloring game, which has been one of the main tools for bounding the game chromatic number, is the *marking game* as introduced by Zhu [17] (see also [1, 5, 12, 13, 14, 16, 18] for some further studies). The marking game can be viewed as the game version of the coloring number, which was introduced by Erdős and Hajnal [9] for infinite graphs. One can make a small modification of the definition from [17] so that it works for infinite graphs. The marking game is played on a graph G by two players, Alice and Bob, who alternate turns in choosing a previously unchosen vertex v of G; at the point v is chosen its score $s(v)$ is determined as the cardinality of the set of (previously) chosen neighbors of v. The resulting invariant, the *game coloring number* is defined as $col_g(G) = 1 + \sup\{s(x) \,|\, x \in V(G)\}$, where it is assumed that Alice's goal is to minimize and Bob's goal is to maximize the final score and both players play optimally. For all concepts mentioned, but not defined in this paper we refer to [7].

The *vertex-edge marking game* has been defined by Bartnicki et al. [2] as a variation of the marking game on vertices: as usual, two players play the game, and while the first player Alice marks vertices, Bob marks edges. The goal of Bob is to surround an unmarked vertex by as many marked edges as possible, while the goal of Alice is opposite; she wishes to keep the number of marked edges incident to an unmarked vertex as small as possible. The *score of a vertex v at a certain state t* of the game, $\mathrm{score}_t(v)$, is the number of marked edges incident with v if v is unmarked at state t, and 0 otherwise. The *score of $v \in V(G)$* is $\mathrm{score}(v) = \sup_t\{\mathrm{score}_t(v)\}$, and the *vertex-edge coloring number* is

$$\mathrm{col}_{ve}(G) = \sup_{v \in V(G)} \{\mathrm{score}(v)\} + 1\,.$$

Note that the above definition of the vertex-edge coloring number is well defined for both finite and infinite graphs and all results we give hold for infinite graphs as well, unless otherwise stated. Also note that the vertex-edge marking game can be extended to multigraphs (allowing multiple edges between two vertices). We begin with the following obvious observation.

Lemma 1. *For every multigraph G, if H is a submultigraph of G, then we have $col_{ve}(H) \leq col_{ve}(G)$.*

The vertex-edge coloring number of a graph G is closely related to the game coloring number of the subdivision $S(G)$ of G, obtained by subdividing every edge of G exactly once:

Proposition 2. *If $col_{ve}(G) > 2$, then $col_{ve}(G) = col_g(S(G))$.*

(The proof of this result is given in Section 5.) Through this connection, one can derive from [13, Example 6.1] that $\mathrm{col}_{ve}(K_{n,n})$ is unbounded when n grows. Consequently, also $\{\mathrm{col}_{ve}(K_n) \,|\, n \in \mathbb{N}\}$ is unbounded, and we give some light on the asymptotic behaviour of the vertex-edge coloring number in complete graphs. More precisely, we prove that

$$\lfloor \log_2(n-1) \rfloor - \lceil \log_2 \lfloor \log_2(n-1) \rfloor \rceil + 2 \leq \mathrm{col}_{ve}(K_n) \leq \lceil \log_2 n \rceil + 2. \tag{1}$$

In fact, the upper bound in (1) holds even if K_n is generalized to $K_n^{(p)}$, where each edge of K_n is replaced by the set of multiple edges of an arbitrarily large cardinality p. For the upper bound we model the game as a process on sorted words of integers that represent the positions in the game.

The paper is organized as follows. In the next section, we prove a basic auxiliary result, which we call the Orientation Lemma, and give several immediate consequences of this result, related to degeneracy and arboricity. In Section 3, we infer from the Orientation Lemma that $\mathrm{col}_{ve}(G) \leq 5$ in finite planar graphs G, and prove sharp upper bounds for the vertex-edge coloring number in cactus graphs and finite outerplanar graphs (which are 3 and 4, respectively). We also determine the exact value of the invariant in the hexagonal lattice and the square lattice, and bound it in the triangular lattice. Section 4 is concerned with complete graphs and the proof of (1). In Section 5 we prove Proposition 2, and in the last section we propose several open problems.

2 Constrained degree orientations

In this section, we present a general upper bound for the vertex-edge coloring number of a graph which involves orientations of its edges. We also give some immediate consequences of this result that will be applied in several proofs of subsequent sections.

A graph has a *d-bounded orientation* if its edges can be oriented in such a way that the maximum out-degree of the vertices in the resulting digraph is at most d. The concept was introduced under this name by Chrobak and Eppstein [6] although similar concepts have been studied earlier (see e.g. [10]).

Lemma 3 (*Orientation Lemma*). *If G is a graph which has a d-bounded orientation, then $col_{ve}(G) \leq d+2$.*

Proof. Consider an oriented digraph D obtained from G by orienting its edges in such a way that $\mathrm{outdeg}(v) \leq d$ for all $v \in V(G)$. Alice's strategy to maintain $\mathrm{score}(v) \leq d+1$ for every vertex v is as follows. Suppose that Bob marked an edge $e = uv$, and $u \to v$ is the orientation of e in D. Then Alice marks v if v has not yet been marked (and otherwise she marks any unmarked vertex). Note that at the point v was marked, uv is the only marked edge in G for which the orientation is towards v. Since v has outdegree at most d, there are thus at most $d+1$ edges incident to v that were already marked at the time v is marked. Hence $\mathrm{score}(v) \leq d+1$, which gives the claimed statement. □

Another well-known parameter related to bounded-degree orientations is degeneracy. A *k-degenerate graph* is a graph in which every subgraph has a vertex incident with at most k edges. Alternatively, a graph is k-degenerate if and only if there is an ordering of its vertices such that every vertex v has at most k backward edges with respect to the ordering. Note that a

k-degenerate graph has a k-bounded orientation. Bounded orientation is also in relation with arboricity: if G can be decomposed into k forests, then it has a k-bounded orientation (simply orient each tree towards a root). Hence, by Lemma 3, we have the following bound.

Corollary 4. *Given a positive integer k, if G is a k-degenerate graph or if its edge-set can be decomposed into k forests, then $col_{ve}(G) \leq k+2$.*

Since $\mathrm{col}_{ve}(P_6) = 3$ we get the following result.

Corollary 5. *For any forest F we have $col_{ve}(F) \leq 3$ and the bound is tight.*

3 Planar graphs

Using the tools of Section 2 and other known results, we derive sharp upper bounds for the vertex-edge coloring number in several classes of planar graphs. In particular, we find exact values of the vertex-edge coloring numbers of two infinite lattices.

First, since every finite planar graph has a 3-bounded orientation (which can even be constructed in linear time, see [6]), we infer by the Orientation Lemma the following general bound for planar graphs.

Proposition 6. *For every finite planar graph we have $col_{ve}(G) \leq 5$.*

We next present an auxiliary result concerning a lower bound for the vertex-edge coloring number, for which we need the following definition. In a graph G on which the vertex-edge marking game is played, a *free-path* is a path P of length at least 3 whose inner vertices have degree at least 3 in G and are unmarked and whose endvertices have incident marked edges in P.

Lemma 7 (Free-path Lemma). *If at some state in the vertex-edge coloring game, a graph G contains a free-path and it is Bob's turn, then Bob has a strategy to force $col_{ve}(G) \geq 4$.*

Proof. Let $P = (x_1, x_2, \ldots, x_{\ell+1})$ be a free-path of length $\ell \geq 3$ in G. We prove the lemma by induction on ℓ. If $\ell = 3$ then Bob marks the edge x_2x_3 (if it is already marked, then he marks an arbitrary unmarked edge), leading to two unmarked vertices x_2 and x_3 with score 2. In the next step Alice can mark at most one vertex from $\{x_2, x_3\}$, thus there exists $i \in \{2, 3\}$ such that x_i is unmarked after that step. Then Bob marks an edge incident with x_i and enforces the score of 3 on x_i. Assume now the lemma is true for free-paths of length at most $\ell' = \ell - 1$. We prove that Bob can ensure a score of 3 with the free-path P of length ℓ. Bob marks the edge x_2x_3 (or an arbitrary unmarked edge if x_2x_3 is already marked). If there is an edge incident to x_2 that is not from P and is already marked, then score$(x_2) \geq 3$ and the proof is completed. Thus we may assume that at this state of the game, the only marked edges incident to x_2 are x_1x_2 and x_2x_3. If Alice in her next step marks x_2, then we have a free-path $(x_2, x_3, \ldots, x_{\ell+1})$ of length

$\ell - 1$ which, by induction, gives the result. Else, if Alice does not mark x_2, then Bob can mark an unmarked edge incident with x_2 to obtain a score of 3 on x_2, which implies $\mathrm{col}_{ve}(G) \geq 4$. □

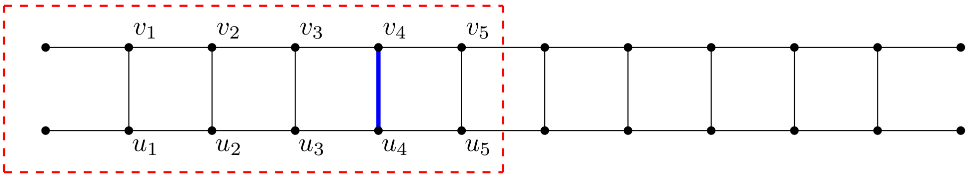

Figure 1: The graph H and one of the two disjoint copies of the graph G (inside the dashed area) contained in H; the thick line indicating the first edge chosen by Bob.

We are now able to prove the following result for outerplanar graphs. The *Cartesian product* $G \,\square\, H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and in which vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Proposition 8. *Let G be a finite outerplanar graph, then $col_{ve}(G) \leq 4$ and the bound is tight.*

Proof. Outerplanar graphs are 2-degenerate, hence Corollary 4 gives the upper bound. For the tightness, we prove that the graph $H = P_{10} \,\square\, P_2$ with a leaf added on each of its four vertices of degree 2 satisfies $\mathrm{col}_{ve}(H) \geq 4$. Let G be the graph $P_5 \,\square\, P_2$ with an added leaf on two adjacent vertices of degree 2. We let u_i and v_i, for $i \in \{1, \ldots, 5\}$ be the vertices of G such that, for $i \in \{1, 2, 3, 4\}$, u_i, v_i, u_{i+1} and v_{i+1} is an induced square of G ($u_i v_i$, $u_{i+1} v_{i+1}$, $u_i u_{i+1}$ and $v_i v_{i+1}$ being the edges of this square). Remark that there are two disjoint copies of G in H (see Figure 1). Consequently, whatever the first vertex marked by Alice, there remains one copy of G unmarked by Alice. The strategy of Bob starts by marking the edge $u_4 v_4$ of this unmarked copy of G in H (see Figure 1).

First, if Alice does not mark u_4 or v_4, then there is at least one edge consisting of unmarked vertices among $u_3 v_3$ and $u_5 v_5$ and Bob marks this edge, say it is $u_3 v_3$. Whatever the vertex Alice marks next, there is a free-path of length 3 between either v_3 and v_4 or between u_3 and u_4, and hence by Lemma 7 there is a vertex of score 3.

Second, if Alice marks u_4 or v_4, then suppose, without loss of generality, that Alice has marked v_4. In this case Bob marks the edge $u_2 v_2$. We distinguish two cases. In the case Alice does not mark u_2 or v_2, then Bob marks an edge containing only non-marked vertices among $u_1 v_1$ and $u_3 v_3$ and whatever the vertex Alice marks, Bob can mark an edge between two unmarked vertices so that they are now both incident with two marked edges. In the case Alice marks u_2, there is a free-path $(v_4, u_4, u_3, v_3, v_2, u_2)$ and thus by Lemma 7, Bob can force a score of at least 3 in some vertex. Otherwise if Alice marks v_2, then there is again a free-path $(v_4, u_4, u_3, u_2, v_2)$, hence, again by Lemma 7, $\mathrm{col}_{ve}(G) \geq 4$. □

A *cactus graph* is a connected graph in which any two simple cycles have at most one vertex in common. Such graphs have a tree structure, i.e., each of its blocks is either a cycle or an edge and the intersection graph of its blocks is a tree. Since every cactus graph G is outerplanar, we have $\mathrm{col}_{ve}(G) \leq 4$ by Proposition 8 for such graphs. However, we prove a stronger bound in the next theorem.

Theorem 9. *For every cactus graph G_C, we have $col_{ve}(G_C) \leq 3$.*

Proof. Assume that G_C has at least one cycle C_1 since otherwise G_C would be a tree and hence Proposition 6 would allow to conclude.

Then in each other block B of G_C, there is a unique vertex x that is closer to C_1 than the other vertices of B (by the tree structure of the cactus). We call this vertex x the *head* of B. For the cycle C_1, we choose an arbitrary vertex to be the head.

The strategy of Alice is the following:

R1. At the beginning, Alice marks any vertex of C_1.

R2. If Bob has marked an edge e that does not lie in a cycle, then if possible, Alice marks the head of e, otherwise (if the head is already marked) Alice marks an arbitrary unmarked vertex of G_C.

R3. If Bob has marked an edge of a cycle C of G_C and no other edges of C are marked, then if possible, Alice marks the head of C, otherwise she marks an arbitrary unmarked vertex of G_C.

R4. Otherwise, if Bob marks an edge $e = uv$ of C and C had already marked edges, then if possible, Alice marks among u and v the vertex that is closer to the first marked edge of C along the path that does not cross the head of C. If this is not possible (the chosen vertex is already marked), then Alice marks an arbitrary vertex of G_C.

We now prove that with this strategy for Alice, there will not exist an unmarked vertex u that is incident with more than two marked edges. If u is not in a cycle, then by Rule R2, only the edge emanating from u towards the root cycle C_1 and at most one edge in the other direction may be marked at the time u is marked. If u lies in a cycle C of G_C and u has both of its two incident edges in C marked, then other edges incident with u are not marked, by Rules R2 and R3. In addition, by Rule R4, Alice will mark u when both of the incident edges of u in C are marked. Otherwise, if at most one edge in C, which is incident with u, is marked, then by Rules R2 and R3, u will be marked as the head of any other edge (or the head of the corresponding cycle) with which u is incident. Hence, in either case, as soon as two edges incident with u are being marked, u will be marked, thus the score of u is at most 2. $\square$

Theorem 10. *If $\mathcal{H}$ is the infinite hexagonal lattice, then $col_{ve}(\mathcal{H}) = 4$.*

Proof. It is easy to orient the edges of $\mathcal{H}$ in such a way that each vertex has out-degree at most 2 (see Figure 2 showing a portion of the hexagonal lattice with the orientation of edges depicted), hence, by Lemma 3, we infer $\mathrm{col}_{ve}(\mathcal{H}) \leq 4$. To prove the lower bound, we are going to show that Bob has a strategy which ensures a score of 3 in some vertex of $\mathcal{H}$.

Consider a sufficiently large portion of the hexagonal lattice such that after Alice's first move, Bob is able to mark an edge $e_1 = x_1y_1$ that is far enough from the vertex marked by Alice (distance 7 should suffice). See Figure 2 for the names of the other edges and vertices considered. If Alice does not mark x_1 or y_1, then Bob marks an edge $e \in \{e_{11}, e_{12}\}$ such that all vertices of the 6-cycle C containing e_1 and e are unmarked. Hence whatever the vertex Alice marks there is a free-path of length 4 on C and by Lemma 7, Bob has a way to force a score of 3.

Now assume, without loss of generality, that Alice has marked vertex x_1. Then Bob marks the edge $e_2 = x_2y_2$. Suppose that in the next move Alice does not mark vertex y_1. Then there remains a free path between either x_2 (if y_2 is not marked) or y_2 (if x_2 is not marked) and x_1 and by Lemma 7, Bob has a way to force a score of 3 in some vertex.

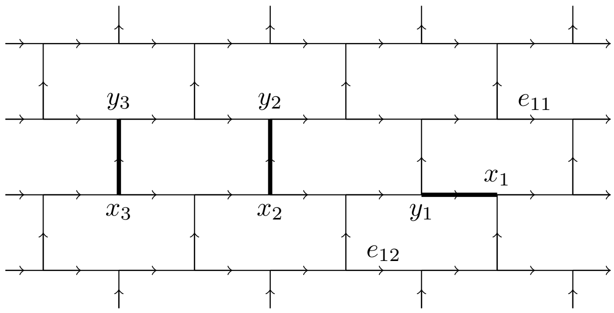

Figure 2: A part of the hexagonal lattice with 2-bounded orientation and some designated edges and vertices.

Otherwise, if Alice marks y_1 then Bob marks the edge $e_3 = x_3y_3$, by which he gets two marked edges of the 6-cycle, none of which vertices is marked. Hence, whatever the vertex marked by Alice, there will remain a free-path between x_2 or y_2 and x_3 or y_3 and thus Bob will be able again to force a score of 3, yielding $\mathrm{col}_{ve}(\mathcal{H}) \geq 4$. $\square$

Proposition 11. *If $\mathcal{S}$ is the infinite square lattice, then $col_{ve}(\mathcal{S}) = 4$.*

Proof. Since the graph H from the proof of Proposition 8 is a subgraph of $\mathcal{S}$, it follows from Lemma 1 that $4 = col_{ve}(H) \leq col_{ve}(\mathcal{S})$.

For the upper bound we use the Orientation Lemma, noting that $\mathcal{S}$ can be oriented in such a way that the out-degree of every vertex of $\mathcal{S}$ is bounded by 2, see Figure 3. $\square$

For the triangular lattice, there is an orientation of its edges such that the out-degree of every vertex is 3 (see Figure 3), hence we infer by Lemma 3 the following upper bound.

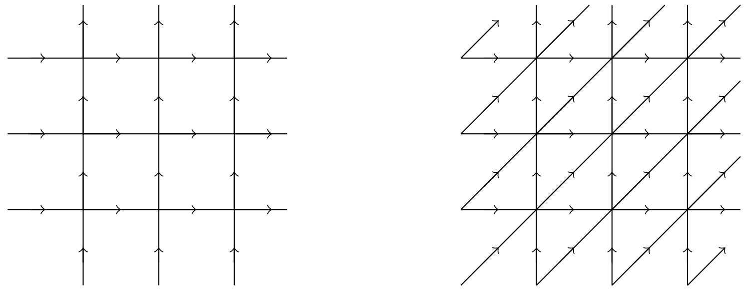

Figure 3: The square lattice with 2-bounded orientation and the triangular lattice with 3-bounded orientation.

Proposition 12. *If $\mathcal{T}$ is the infinite triangular lattice, then $col_{ve}(\mathcal{T}) \leq 5$.*

Since $\mathcal{H}$ is a spanning subgraph of $\mathcal{T}$, we infer the lower bound $\mathrm{col}_{ve}(\mathcal{T}) \geq 4$. We wonder what is the exact value of the vertex-edge coloring number of $\mathcal{T}$. The question is related also to the exact upper bound of this number in planar graphs.

4 Complete graphs

In order to prove the upper bound on the vertex-edge marking game on complete graphs, we find convenient to model the game as a process on sorted words of integers that will represent the positions of the game (i.e., the number of incident marked edges of each unmarked vertex). This model enables us to prove the upper bound for a family of multigraphs that generalize complete graphs; notably, given a positive integer n and a non-zero cardinal number p, the multigraph $K_n^{(p)}$ has n vertices and between each pair of vertices there are p parallel edges.

We first introduce some notation and two lemmas on sorted words. Let S be a finite sequence of non-negative integers in non-increasing order, i.e., let $S = s_1 s_2 \cdots s_p$ be a word over the alphabet of integers, with $s_1 \geq s_2 \geq \cdots \geq s_p$; $p = |S|$. This is called a *sorted word*. We consider the process that starts from a sorted word S and apply inductively the operation that consists in suppressing the first letter s_1 of the word and adding the value 1 to two distinct letters of the word (and then reordering the letters of the word in such a way that it becomes a sorted word). More formally, let f be a function that maps a sorted word $S = s_1 \cdots s_p$, where $p \geq 3$, to a sorted word $T = t_1 \cdots t_{p-1}$, such that there exist $i, j \in \{2, \ldots, p\}$, $i \neq j$, and $s'_k = s_{k+1}$ for $j \neq k+1 \neq i$, and $s'_{i-1} = s_i + 1$, $s'_{j-1} = s_j + 1$, and T is obtained from $S' = s'_1 s'_2 \cdots s'_{p-1}$ by an eventual reordering of S' to create a non-increasing order. We then write $f(S) = T$. When $p = 2$, we let $f(s_1 s_2) = s_2 + 1$ (that is, the resulting word T is of length 1).

Let $M(S)$ be the maximum integer in a word that can be obtained by this process starting from the word S. More formally, let $m(S)$ be the maximum (i.e., the first) integer of a word S. Then let $\mathcal{S}_S = \{S' |\ \exists f_1, \ldots, f_k : S' = f_k \circ \ldots \circ f_1(S)\}$. Alternatively, we can define $M(S)$ as

$\max\{m(S')|\ S' \in \mathcal{S}_S\}$.

We also define the partial order $\preceq$ on sorted words by $S \preceq S'$ if $|S| \leq |S'|$ and for each i, $1 \leq i \leq |S|$, $s_i \leq s'_i$.

Lemma 13. *The following properties hold for sorted words.*

(i) if $S \preceq S'$, then $M(S) \leq M(S')$;

(ii) if $f(S) = S'$, then $M(S') \leq M(S)$;

(iii) for any positive integer i, $M(1^i) = 1 + M(0^i)$;

(iv) if $M(S') > s'_1$, $|S| = |S'|$, and $s_i \geq s'_i$ for every integer i, $i \geq 2$, then $M(S) \geq M(S')$.

(v) for any integers $r \geq 2$ and $s \geq 2$, we have $M(1^r0^s) \leq M(1^{r+1}0^{s-2})$.

Proof. (i) Let S_m be the word of $\mathcal{S}_S$ such that $m(S_m) = M(S)$. Let $f_1, \ldots, f_k$ be the functions such that $S_m = f_k \circ \ldots \circ f_1(S)$. Since $S \preceq S'$, then clearly, for the word $S'_m = f_k \circ \ldots \circ f_1(S')$ we have $m(S'_m) \geq m(S_m) = M(S)$.
(ii) If $f(S) = S'$, then $\mathcal{S}_{S'} \subseteq \mathcal{S}_S$, which implies that $M(S') \leq M(S)$.
(iii) Let S_m be a sorted word of $\mathcal{S}_{0^i}$ such that $m(S_m) = M(0^i)$. The same sequence of functions used to obtain S_m from 0^i can be used on 1^i to obtain a sorted word S'_m such that $m(S'_m) = M(1^i)$. We infer $m(S'_m) = m(S_m) + 1$.
(iv) Let S'_m be the word of $\mathcal{S}_{S'}$ such that $m(S'_m) = M(S')$. Since $M(S') > s'_1$, there exist the set of functions $f_1, \ldots, f_k$ such that $S'_m = f_k \circ \ldots \circ f_1(S')$. Note that $f_k \circ \ldots \circ f_1(S)$ yields a sorted word S_m such that $M(S) \geq m(S_m) \geq m(S'_m) = M(S')$.
(v) Let us prove that for any integers $r \geq 2$ and $s \geq 2$, we have $M(1^r0^s) \leq M(1^{r+1}0^{s-2})$. Let $1^r0^s = P_0, P_1, \ldots, P_n = S_m$ be a sequence of words with $f_i(P_i) = P_{i+1}$ for $0 \leq i \leq n-1$, and $M(1^r0^s) = m(S_m)$. Let t be the smallest integer such that P_t does not contain the subword 0^s. (If such an integer t does not exists, then we have $M(1^r0^s) = M(1^r) \leq M(1^{r+1}0^{s-2})$, by property (i), as desired.) Let $Q_1 = 1^{r+1}0^{s-2}$. For any i, $2 \leq i \leq t$, we denote by Q_i the sorted word $f_{i-2} \circ \cdots f_0(Q_1)$. See the diagram on Figure 4.

$$P_0 = 1^r0^s \xrightarrow[f_0]{\qquad} P_1 = f_0(1^r0^s) \overset{}{\underset{f_1 \cdots f_{t-2}}{\dashrightarrow}} P_{t-1} = f_{t-2} \circ \cdots \circ f_0(1^r0^s)$$

$$Q_1 = 1^{r+1}0^{s-2} \xrightarrow[f_0]{\qquad} Q_2 = f_0(1^{r+1}0^{s-2}) \underset{f_1 \cdots f_{t-2}}{\dashrightarrow} Q_t = f_{t-2} \circ \cdots \circ f_0(1^{r+1}0^{s-2})$$

Figure 4: Illustration of the proof of Lemma 13.

Note that $P_{t-1} = f_{t-2} \circ \cdots \circ f_0(1^r0^s) = u_1 \ldots u_{r-t+1}0^s$, and $Q_t = f_{t-2} \circ \cdots \circ f_0(1^{r+1}0^{s-2}) = u_1 \ldots u_{r-t+1}1^10^{s-2}$ (since $f_1, \ldots f_{t-2}$ are functions whose composition changes the r first integers equal to 1 into $u_1 \dots u_{r-t+1}$). Observe that P_t is obtained from P_{t-1} by deleting the first integer and adding 1 to two integers, at least one of which is 0.

We distinguish two cases. If P_t was obtained from P_{t-1} by adding 1 to two integers equal to 0, then $P_t = u_2 \dots u_{r-t+1}1^2 0^{s-2}$, and clearly $P_t \preceq Q_t$, which implies $M(P_t) \le M(Q_t)$ by property (i). Thus, we have $M(1^r 0^s) = M(P_t) \le M(Q_t) \le M(1^{r+1}0^{s-2})$.

The second case is that P_t is obtained from P_{t-1} by suppressing u_1, changing one integer 0 to 1 and increasing by 1 an integer u_i, where $i \in \{2, \dots, r-t+1\}$. Suppose $u_i < u_2$. Then, let j be the largest index in $\{2, \dots, i-1\}$ such that $u_j > u_i$. Hence,

$$P_t = u_2 \dots u_j(u_i+1)u_{j+1} \dots u_{i-1}u_{i+1} \dots u_{r-t+1}1^1 0^{s-1}.$$

Note that $P_t \preceq Q_t$, since $u_j \ge u_i + 1$. We again derive in the same way that $M(1^r 0^s) \le M(1^{r+1}0^{s-2})$. Finally, suppose that $u_i = u_2$. Then $P_t = (u_2+1)u_3 \dots u_{r-t+1}10^{s-1}$. If $M(P_0) = M(P_t) = u_2 + 1$, then $M(Q_t) \ge u_2 + 1$, because for $Q' = f(Q_t)$, where f is a function that increases the second integer by 1, we get $M(Q_t) \ge M(Q') \ge u_2+1$. Otherwise, if $M(P_t) > u_2+1$, then by property (iv) applied on Q_t and P_t, we get $M(Q_t) \ge M(P_t)$. In either case, we infer $M(1^r 0^s) = M(P_t) \le M(Q_t) \le M(1^{r+1}0^{s-2})$. □

The properties of Lemma 13 are used to prove the following lemma.

Lemma 14. *For any $k \ge 1$ we have $M(0^{2^k+1}) = k + 1 = \lceil \log_2(2^k+1) \rceil$.*

Proof. Let $t = 2^k + 1$, where $k \ge 1$. Using the sequence of functions that always change two zeros, we get the sequence of words, $0^t, 1^2 0^{t-3}, \dots, 1^{\lceil \frac{t}{2} \rceil - 1}0^2, 1^{\lceil \frac{t}{2} \rceil}$. Therefore $M(1^{\lceil \frac{t}{2} \rceil}) \le M(0^t)$.

Now, we show the reversed inequality, $M(0^t) \le M(1^{\lceil \frac{t}{2} \rceil})$. Since $\mathcal{S}_{0^t} = \mathcal{S}_{1^2 0^{t-3}} \cup \{0^t\}$, we have $M(0^t) = M(1^2 0^{t-3})$. If $k = 1$, that is, $t = 3$, this gives $M(0^t) = M(0^3) = M(1^2) = M(1^{\lceil \frac{t}{2} \rceil})$. Otherwise, we apply Lemma 13(v) several times and we get $M(0^t) = M(1^2 0^{t-3}) \le M(1^3 0^{t-5}) \le \dots \le M(1^{\lceil \frac{t}{2} \rceil})$.

We prove $M(0^{2^k+1}) = k+1$ by induction on k. When $k = 1$, we clearly have $M(000) = 2$. By the above, $M(0^{2^k+1}) = M(1^{2^{k-1}+1})$. By Lemma 13(iii), this is in turn equal to $1 + M(0^{2^{k-1}+1})$, which is by induction equal to k. Hence, $M(0^{2^k+1}) = k + 1$. □

Corollary 15. *For any $n \ge 3$, we have $M(0^n) \le \lceil \log_2 n \rceil + 1$.*

Proof. For $n \ge 3$, let k be the integer such that $2^{k-1} + 1 < n \le 2^k + 1$. Thus, by Lemma 13(i) and Lemma 14, $M(0^n) \le M(0^{2^k+1}) = k + 1 \le \lceil \log_2 n \rceil + 1$. □

We think that in fact $M(0^n) = \lceil \log_2 n \rceil$, but could not improve the upper bound from Corollary 15. We are now ready to prove the upper bound for $\mathrm{col}_{ve}(K_n^{(p)})$.

Theorem 16. *For a non-zero cardinal number p and every $n \ge 2$,*

$$col_{ve}(K_n^{(p)}) \le \lceil \log_2 n \rceil + 2.$$

Proof. We are going to prove the upper bound $\lceil \log_2 n \rceil + 2$ for the multigraph $K_n^{(p)}$, where $p \geq n-1$. Then, Lemma 1 yields the statement of the theorem for any $p < n-1$ as well.

The strategy of Alice is to mark at each step a vertex having a maximum number of incident edges that are marked. We will prove that whatever the strategy of Bob, there will be no unmarked vertex with more than $\lceil \log_2 n \rceil + 1$ incident marked edges. Clearly, at any step for which there remain at least two unmarked vertices, we can assume that Bob marks an edge $e = xy$ with both x and y being not already marked, by which the score of two vertices is increased. When just one vertex x remains unmarked by Alice, then Bob marks an edge incident with x increasing its score by 1 (therefore, before the penultimate move of Alice, x and y have been unmarked, and if they have the same number of incident marked edges at that time, then it is possible that $\mathrm{col}_{ve}(K_n^{(p)})$ is attained only by the score of x). With this hypothesis, we can represent the game by a sequence $S_0, S_1, \ldots, S_{n-1}$ of sorted words of integers as described above; word S_i, where $0 \leq i \leq n-1$, is the sorted word that contains the numbers of marked edges incident with each unmarked vertex of $K_n^{(p)}$ after the ith move of Bob. In addition, S_i is obtained from S_{i-1} by a function as described above. Since we have $S_0 = 0^n$, then, by virtue of Corollary 15, we obtain $M(S_0) \leq \lceil \log_2 n \rceil + 1$, and hence $\mathrm{col}_{ve}(K_n^{(p)}) \leq \lceil \log_2 n \rceil + 2$. □

Clearly, plugging $p = 1$ in Theorem 16, we get $\mathrm{col}_{ve}(K_n) \leq \lceil \log_2 n \rceil + 2$. Note that Lemma 14 implies that $\mathrm{col}_{ve}(K_n) \leq \lceil \log_2 n \rceil + 1$, for $n = 2^k + 1$.

Now, we prove the lower bound in (1) by presenting a strategy for Bob for which at least one vertex will have a score of $\lfloor \log_2(n-1) \rfloor - \lceil \log_2 \lfloor \log_2(n-1) \rfloor \rceil + 1$ whatever Alice's strategy.

Theorem 17. *For every $n \geq 3$, we have*

$$col_{ve}(K_n) \geq \lfloor \log_2(n-1) \rfloor - \lceil \log_2 \lfloor \log_2(n-1) \rfloor \rceil + 2.$$

Proof. First, we consider the graph K_n, where $n = 2^k + 1$. The strategy of Bob consists of several steps, in each of which Bob marks edges of a matching. After the ith step Bob can ensure that there exists a subgraph G_i with 2^{k-i} unmarked vertices each of which is incident with i marked edges.

Note that Alice starts the game by marking an arbitrary vertex x. Let $X_0 = \{x\}$ and $G_0 = K_n - X_0$. Clearly, G_0 has 2^k unmarked vertices each of which is incident with 0 marked edges (which presents the zero-th step).

We follow with the first step and it is Bob's turn. In the next 2^{k-1} moves Bob marks edges of a perfect matching of G_0. During this time, Alice marks 2^{k-1} vertices (denote this set of vertices by X_1) of G_0. Let $G_1 = G_0 - X_1$, and note that G_1 has (at least) 2^{k-1} unmarked vertices each of which is incident with 1 marked edge. This ends the first step and note that Alice was the last to play in this step.

In the ith step we note by induction that there exists a subgraph G_{i-1} with 2^{k-i+1} unmarked vertices each of which is incident with $i-1$ marked edges. If there exists a perfect matching in

G_{i-1} that consists of non-marked edges, then in the next 2^{k-i} moves Bob marks edges of this perfect matching. During this time, Alice marks 2^{k-i} vertices (denote this set of vertices by X_i) of G_{i-1}. Then $G_i = G_{i-1} - X_i$ has (at least) 2^{k-i} unmarked vertices each of which is incident with i marked edges. We apply Dirac's theorem [8], which ensures a Hamiltonian cycle in a graph H with even order if each vertex has degree at least half of the order. This in turn implies the existence of a perfect matching in H. Therefore, Bob can ensure the existence of a perfect matching of non-marked edges in G_{i-1} if

$$|V(G_{i-1})| - 1 - (i-1) \geq \frac{1}{2}|V(G_{i-1})|.$$

That is, $2^{k-i+1} - i \geq 2^{k-i}$, which gives

$$2^{k-i} \geq i. \tag{2}$$

The number of steps (in Bob's strategy) is the largest i such that (2) is fulfilled. When this condition is no longer fulfilled (after the ith step), Bob can mark an edge incident to an unmarked vertex of G_i by which the score of this vertex is at least $i+1$ (and so $col_{ve}(K_n) \geq i+2$).

Since $i \in \mathbb{N}$, the largest i satisfying (2) is $k - \lceil \log_2 k \rceil$ or $k - \lceil \log_2 k \rceil + 1$ (depending on k). In the case $n = 2^k + 1$, we get $col_{ve}(K_n) \geq \log_2(n-1) - \lceil \log_2(\log_2(n-1)) \rceil + 2$.

Finally, let $2^k \leq n-1 < 2^{k+1}$. Therefore,

$$col_{ve}(K_n) \geq col_{ve}(K_{2^k+1}) \geq k - \lceil \log_2 k \rceil + 2 \geq \lfloor \log_2(n-1) \rfloor - \lceil \log_2 \lfloor \log_2(n-1) \rfloor \rceil + 2.$$

□

5 Relations with the marking game

We will prove that the vertex-edge coloring number of a graph G coincides with the game coloring number of the graph $S(G)$ obtained from G by subdividing all of its edges once, as soon as the vertex-edge coloring number of G is at least 3 (Proposition 2).

First, we prove that the class of graphs G with $\mathrm{col}_{ve}(G) \leq 2$ is small. Clearly, only graphs with no edges have this number equal to 1. We characterize the graphs with $\mathrm{col}_{ve}(G) = 2$ as follows.

Proposition 18. *If G is a non-empty graph, then $col_{ve}(G) = 2$ if and only if G is a forest with at most one connected component of diameter at most 4 and all other connected components of diameter at most 2.*

Proof. First, suppose that $\mathrm{col}_{ve}(G) = 2$. Remark that if Bob can mark an edge having each of its end-vertices unmarked and incident to an unmarked edge, then it implies $\mathrm{col}_{ve}(G) \geq 3$. If G contains a cycle, such an edge can be found in Bob's first move. Thus G is a forest. If G has a connected component T of diameter at least 5 or if G contains two connected components both having diameter at least 3, such an edge can be found at first Bob's move whatever the vertex Alice has chosen in her first move.

For the converse, let G be a non-empty forest with at most one connected component T_1 of diameter at most 4 and all other connected components $T_2, \ldots, T_k$ of diameter at most 2. Let c_i be a center of T_i for any $i \in \{1, \ldots, k\}$. The strategy of Alice is to first mark c_1 and then after each Bob's move (in which he marks an edge e), she marks (if possible) an unmarked vertex incident with e that is not a leaf. Therefore the score of each vertex in G is at most 1 and $\mathrm{col}_{ve}(G) = 2$. □

For the purpose of proving the next result, which connects the vertex-edge marking game on a graph G with the (standard) marking game on the subdivided graph $S(G)$, we propose two variations of the vertex-edge marking game. In the first variation, which we call *vertex-edge-star-Alice marking game*, Alice is allowed to play also on the edges while Bob's role does not change. The corresponding score of the game will be denoted by $\mathrm{col}^{*A}_{ve}(G)$, and is defined exactly the same as in the standard vertex-edge marking game, that is, $\sup\limits_{v \in V(G)} \{\mathrm{score}(v)\} + 1$. As in the vertex-edge marking game, $\mathrm{score}(v) = \sup_t\{\mathrm{score}_t(v)\}$, where score_t is the number of marked edges surrounding the vertex v at state t if v is unmarked, and 0 if v is marked at state t. Since Alice may choose to play on the vertices of G as long as possible also in the vertex-edge-star-Alice marking game, and it is not to her advantage to play on the edges, it is clear that the new invariant gives the same score.

Similarly, we call *vertex-edge-star-Bob marking game* the game in which Bob is allowed to play also on the vertices while Alice's role does not change. The corresponding score of the game will be denoted by $\mathrm{col}^{*B}_{ve}(G)$, and is again defined in the same way as above. Since Bob may choose to play on the edges of G as long as possible in this version of the vertex-edge marking game, and it is not to his advantage to play on the vertices, we get the following observation.

Lemma 19. *For any graph G, $col^{*A}_{ve}(G) = col_{ve}(G) = col^{*B}_{ve}(G)$.*

We are now able to prove Proposition 2.

Proof of Proposition 2. Consider a strategy of Alice played in the vertex-edge-star-Bob marking game on G, which bounds $\mathrm{score}(v)$ from above by $\mathrm{col}^{*B}_{ve}(G)-1$ for all vertices v of G. Alice can use the same strategy in the marking game in the graph $S(G)$ by playing only on the original vertices of G. During the marking game on $S(G)$, she will imagine a vertex-edge-star-Bob marking game be played on G, and will copy her moves from the optimal strategy on the vertex-edge-star-Bob marking game on G to the real game played on $S(G)$. Note that the resulting score $s(v)$ of a vertex v is bounded by $\mathrm{col}^{*B}_{ve}(G)-1$ if Alice plays optimally (since the score of subdivided vertices is at most 3, the maximum score will be achieved by an original vertex except possibly when $\mathrm{col}_{ve}(G) = 3$). This gives $\mathrm{col}_g(S(G)) \leq col^{*B}_{ve}(G)$.

To see the reversed inequality, let us consider a strategy of Bob in the vertex-edge-star-Alice marking game, which ensures that $\sup_{v\in V(G)}\{\text{score}(v)\}$ of a vertex v in G is at least $\text{col}_{ve}^{*A}(G)-1$. While playing the marking game on $S(G)$, Bob uses this strategy in the vertex-edge-star-Alice marking game on G, by playing the subdivided vertices of $S(G)$ in the corresponding order. In this way, $\sup\{s(v)\,|\,v\in V(G)\}\geq \text{col}_{ve}^{*A}(G)-1$, which gives $\text{col}_g(S(G))\geq \text{col}_{ve}^{*A}(G)$. By Lemma 19, the proof follows. $\square$

The above result implies that the results in this paper for the vertex edge coloring number of a graph G yield the same results for the game coloring number of the subdivision graph $S(G)$, as soon as G is not a forest with at most one connected component of diameter at most 4 and all other connected components of diameter at most 2. (Otherwise, one can check that, for instance, $\text{col}_{ve}(P_3)=2$, yet $\text{col}_g(S(P_3))=\text{col}_g(P_5)=3$.) Combining Proposition 2 with Theorem 9 we can improve the general upper bound 5 for the game coloring number of cactus graphs given by Sidorowicz [15] to the value 3 in the special case of subdivided cactus graphs.

6 Concluding remarks

There are a number of well studied classes of graphs for which it would be interesting to establish whether the vertex-edge coloring number is bounded by a constant. (Clearly, if a class of graphs is k-degenerate for some fixed k, then Corollary 4 provides a positive answer.) In particular, we propose to consider the class of hypercubes, and pose the following

Question 1. *Is $\{col_{ve}(Q_n)\,|\,n\in\mathbb{N}\}$, where Q_n denotes the hypercube of dimension n, bounded by a constant?*

As proven in Section 3, finite planar graphs admit a general upper bound of 5 for their vertex-edge coloring number. There are several examples of (finite or infinite) planar graphs G with $\text{col}_{ve}(G)=4$, so we wonder what is the correct sharp bound in planar graphs. We thus pose the following question.

Question 2. *Is there a (finite) planar graph G with $col_{ve}(G)=5$?*

(Note that for infinite planar graphs we did not establish a general upper bound for the vertex-edge marking game.) It seems that a good candidate for which Question 2 could have an affirmative answer is the triangular lattice.

Question 3. *Is $col_{ve}(\mathcal{T})$ for the triangular lattice $\mathcal{T}$ equal to 4 or 5?*

Trivially, the following general upper bound $\text{col}_{ve}(G)\leq\Delta(G)+1$ holds in every graph G. Note that $\text{col}_{ve}(C_n)=3$ for any $n\geq 3$, hence the bound is attained in cycles, as well as in the hexagonal lattice, since $\text{col}_{ve}(\mathcal{H})=4$. We propose the problem of characterizing the graphs G in which $\text{col}_{ve}(G)=\Delta(G)+1$, and pose the question about the most interesting case.

Question 4. *For which graphs G with maximum degree 3 we have $col_{ve}(G) = 4$?*

A logarithmic upper bound for complete graphs, see Theorem 16, suggests that in many classes of finite graphs the vertex-edge coloring number is bounded by a constant. Therefore, it would be interesting to find a graph operation by which one could built a family of finite graphs with unbounded vertex-edge coloring number. We think that the lexicographic product of graphs could be such an operation. Let G and H be finite graphs. The *lexicographic product* $G \circ H$ of G and H has $V(G \circ H) = V(G) \times V(H)$, and $(g, h)(g', h') \in E(G \circ H)$ if either $g = g'$ and $hh \in E(H)$, or $gg' \in E(G)$. We propose the following question for which we suspect it has an affirmative answer.

Question 5. *Is it true that $col_{ve}(G \circ K_4) \geq col_{ve}(G) + 1$? More generally, is $col_{ve}(G \circ K_{2^n+1}) \geq col_{ve}(G) + n$?*

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